Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2010 Fermat Contest

(Grade 11)

Thursday, February 25, 2010

Solutions

1. The sum consists of two halves and three thirds, each of which equals a whole. Therefore, the sum is 2.

   ANSWER: (A)

2. The quantity 2% is equivalent to the fraction $\frac{2}{100}$, so "2% of 1" is equal to $\frac{2}{100}$.

   ANSWER: (A)

3. *Solution 1*
   Since $PQ = 1$ and $QR = 2PQ$, then $QR = 2$.
   Since $QR = 2$ and $RS = 3QR$, then $RS = 3(2) = 6$.
   Therefore, $PS = PQ + QR + RS = 1 + 2 + 6 = 9$.

   Solution 2
   From the given information,

   $$PS = PQ + QR + RS = PQ + QR + 3QR = PQ + 4QR = PQ + 4(2PQ) = 9PQ$$

   Thus, $PS = 9(1) = 9$.

   ANSWER: (C)

4. Substituting, $x = \frac{1}{3}(3 - 4u) = \frac{1}{3}(3 - 4(-6)) = \frac{1}{3}(3 + 24) = \frac{1}{3}(27) = 9$.

   ANSWER: (C)

5. *Solution 1*
   Since $2^x = 16$, then $2^{x+3} = 2^3 2^x = 8(16) = 128$.

   Solution 2
   Since $2^x = 16$ and $2^4 = 16$, then $x = 4$.
   Since $x = 4$, then $2^{x+3} = 2^7 = 128$.

   ANSWER: (D)

6. A 12 by 12 grid of squares will have 11 interior vertical lines and 11 interior horizontal lines. (In the given 4 by 4 example, there are 3 interior vertical lines and 3 interior horizontal lines.)
   Each of the 11 interior vertical lines intersects each of the 11 interior horizontal lines and creates an interior intersection point.
   Thus, each interior vertical line accounts for 11 intersection points.
   Therefore, the number of interior intersection points is $11 \times 11 = 121$.

   ANSWER: (B)

7. Since PQS is a straight line and $\angle PQR = 110^\circ$, then $\angle RQS = 180^\circ - \angle PQR = 70^\circ$.
   Since the sum of the angles in $\triangle QRS$ is 180°, then

   $$\begin{aligned} 70^\circ + (3x)^\circ + (x + 14)^\circ &= 180^\circ \\ 70 + 3x + x + 14 &= 180 \\ 4x &= 96 \\ x &= 24 \end{aligned}$$

   ANSWER: (D)

8. Each of the vertical strips accounts for $\frac{1}{2}$ of the total area of the rectangle.
The left strip is divided into three equal pieces, so $\frac{2}{3}$ of the left strip is shaded, accounting for $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ of the large rectangle.
The right strip is divided into four equal pieces, so $\frac{2}{4} = \frac{1}{2}$ of the right strip is shaded, accounting for $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of the large rectangle.
Therefore, the total fraction of the rectangle that is shaded is $\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$.
Answer: (E)

9. From the definition, $(5\nabla 1) + (4\nabla 1) = 5(5-1) + 4(4-1) = 5(4) + 4(3) = 20 + 12 = 32$.
Answer: (D)

10. Since $2x^2 = 9x - 4$, then $2x^2 - 9x + 4 = 0$.
Factoring, we obtain $(2x-1)(x-4) = 0$.
Thus, $2x = 1$ or $x = 4$.
Since $x \neq 4$, then $2x = 1$.
Answer: (B)

11. Since the coins in the bag of loonies are worth \$400, then there are 400 coins in the bag.
Since 1 loonie has the same mass as 4 dimes, then 400 loonies have the same mass as 4(400) or 1600 dimes.
Therefore, the bag of dimes contains 1600 dimes, and so the coins in this bag are worth \$160.
Answer: (C)

12. Suppose that each of the 7 people received q candies under the first distribution scheme.
Then the people received a total of $7q$ candies and 3 candies were left over. Since there were k candies, then $k = 7q + 3$.
Multiplying both sides by 3, we obtain $3k = 21q + 9$.
When $21q + 9$ candies were distributed to 7 people, each person could have received $3q + 1$ candies, accounting for $21q + 7$ candies in total, with 2 candies left over. (The 7 people could not each receive more than $3q+1$ candies since this would account for at least $7(3q+2) = 21q+14$ candies, which is too many in total.)
Therefore, 2 candies would be left over.
Answer: (B)

13. If 50 numbers have an average of 76, then the sum of these 50 numbers is $50(76) = 3800$.
If 40 numbers have an average of 80, then the sum of these 40 numbers is $40(80) = 3200$.
Therefore, the sum of the 10 remaining numbers is $3800 - 3200 = 600$, and so the average of the 10 remaining numbers is $\frac{600}{10} = 60$.
Answer: (A)

14. Choice (A) is not necessarily true, since the four friends could have caught 2, 3, 3, and 3 fish.
Choice (B) is not necessarily true, since the four friends could have caught 1, 1, 1, and 8 fish.
Choice (C) is not necessarily true, since the four friends could have caught 2, 3, 3, and 3 fish.
Choice (E) is not necessarily true, since the four friends could have caught 1, 1, 1, and 8 fish.
Therefore, choice (D) must be the one that must be true.
We can confirm this by noting that it is impossible for each of the four friends to have caught at least 3 fish, since this would be at least 12 fish in total and they only caught 11 fish.
Answer: (D)

15. If $-1 < \sqrt{p} - \sqrt{100} < 1$, then $-1 < \sqrt{p} - 10 < 1$ or $9 < \sqrt{p} < 11$.
Since $\sqrt{p}$ is greater than 9, then p is greater than $9^2 = 81$.
Since $\sqrt{p}$ is less than 11, then p is less than $11^2 = 121$.
In other words, $81 < p < 121$.
Since p is a positive integer, then $82 \le p \le 120$.
Therefore, there are $120 - 82 + 1 = 39$ such integers p.

Answer: (D)

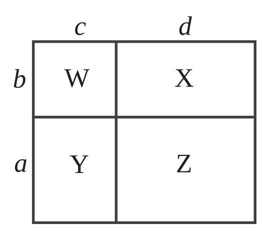
16. Note that $2010 = 10(201) = 2(5)(3)(67)$ and that 67 is prime.
Therefore, the positive divisors of 2010 are 1, 2, 3, 5, 6, 10, 15, 30, 67, 134, 201, 335, 402, 670, 1005, 2010.
Thus, the possible pairs (a, b) with $ab = 2010$ and $a > b$ are $(2010, 1)$, $(1005, 2)$, $(670, 3)$, $(402, 5)$, $(335, 6)$, $(201, 10)$, $(134, 15)$, $(67, 30)$.
Of these pairs, the one with the smallest possible value of $a - b$ is $(a, b) = (67, 30)$, which gives $a - b = 37$.

Answer: (A)

17. Since $PQRS$ is a rectangle, then PQ is perpendicular to QR.
Therefore, the area of $\triangle PQR$ is $\frac{1}{2}(PQ)(QR) = \frac{1}{2}(5)(3) = \frac{15}{2}$.
Since $PT = TU = UR$, then the areas of $\triangle PTQ$, $\triangle TUQ$ and $\triangle URQ$ are equal. (These triangles have bases PT, TU and UR of equal length and each has height equal to the distance between Q and line segment PR.)
Therefore, the area of $\triangle TUQ$ is $\frac{1}{3}(\frac{15}{2}) = \frac{5}{2}$.
Similarly, the area of $\triangle TUS$ is $\frac{5}{2}$.
The area of quadrilateral $STQU$ is the sum of the areas of $\triangle TUQ$ and $\triangle TUS$, or $\frac{5}{2} + \frac{5}{2} = 5$.

Answer: (B)

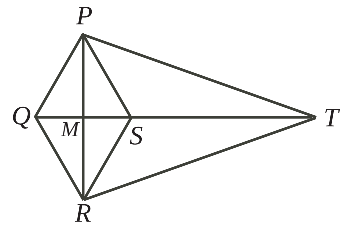
18. Label the lengths of the vertical and horizontal segments as a, b, c, d, as shown.



Rectangle W is b by c, so its perimeter is $2b + 2c$, which equals 2.
Rectangle X is b by d, so its perimeter is $2b + 2d$, which equals 3.
Rectangle Y is a by c, so its perimeter is $2a + 2c$, which equals 5.
Rectangle Z is a by d, so its perimeter is $2a + 2d$.
Therefore, $2a+2d = (2a+2b+2c+2d)-(2b+2c) = (2a+2c)+(2b+2d)-(2b+2c) = 5+3-2 = 6$.

Answer: (A)

19. *Solution 1*
First, we note that $\triangle PQS$ and $\triangle RQS$ are equilateral.
Join P to R. Since $PQRS$ is a rhombus, then PR and QS bisect each other at their point of intersection, M, and are perpendicular.
Note that $QM = MS = \frac{1}{2}QS = 3$.

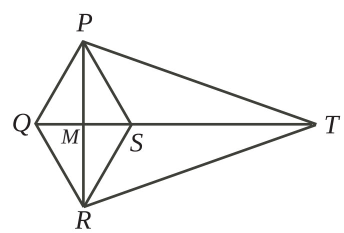


Since $\angle PSQ = 60^\circ$, then $PM = PS\sin(\angle PSM) = 6\sin(60^\circ) = 6(\frac{\sqrt{3}}{2}) = 3\sqrt{3}$.
Since $PT = TR$, then $\triangle PRT$ is isosceles.
Since M is the midpoint of PR, then TM is perpendicular to PR.
Since SM is also perpendicular to PR, then S lies on TM.
By the Pythagorean Theorem in $\triangle PMT$, since $MT > 0$, we have

$$MT = \sqrt{PT^2 - PM^2} = \sqrt{14^2 - (3\sqrt{3})^2} = \sqrt{196 - 27} = \sqrt{169} = 13$$

Therefore, $ST = MT - MS = 13 - 3 = 10$.

Solution 2
First, we note that $\triangle PQS$ and $\triangle RQS$ are equilateral.
Join P to R. Since $PQRS$ is a rhombus, then PR and QS bisect each other at their point of intersection, M, and are perpendicular.
Since $PT = TR$, then $\triangle PRT$ is isosceles.
Since M is the midpoint of PR, then TM is perpendicular to PR.
Since SM is also perpendicular to PR, then S lies on TM.



Since $\angle PSQ = 60^\circ$, then $\angle PST = 180^\circ - 60^\circ = 120^\circ$.
Therefore, in $\triangle PST$, we know that $\angle PST = 120^\circ$, that $PS = 6$ and that $PT = 14$.
By the cosine law,

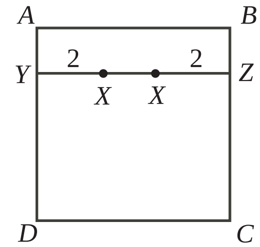
$$\begin{aligned}
PT^2 &= PS^2 + ST^2 - 2(PS)(ST)\cos(\angle PST) \\
14^2 &= 6^2 + ST^2 - 2(6)(ST)\cos(120^\circ) \\
196 &= 36 + ST^2 + 6ST \qquad (\text{since } \cos(120^\circ) = -\tfrac{1}{2}) \\
0 &= ST^2 + 6ST - 160 \\
0 &= (ST - 10)(ST + 16)
\end{aligned}$$

and so $ST = 10$ or $ST = -16$. Since $ST > 0$, then $ST = 10$.

Answer: (D)

20. Label the square as $ABCD$.
Suppose that the point X is 1 unit from side AB.
Then X lies on a line segment YZ that is 1 unit below side AB.

Note that if X lies on YZ, then it is automatically 4 units from side DC.
Since X must be 2 units from either side AD or side BC, then there are 2 possible locations for X on this line segment:



Note that in either case, X is 3 units from the fourth side, so the four distances are 1, 2, 3, 4 as required.
We can repeat the process with X being 2, 3 or 4 units away from side AB. In each case, there will be 2 possible locations for X.
Overall, there are $4(2) = 8$ possible locations for X. These 8 locations are all different, since there are 2 different points on each of 4 parallel lines.

ANSWER: (D)

21. *Solution 1*
Since the problem asks us to find *the* value of $\frac{x-z}{y-z}$, then this value must be the same no matter what x, y and z we choose that satisfy $\frac{x-y}{z-y} = -10$.
Thus, if we can find numbers x, y and z that give $\frac{x-y}{z-y} = -10$, then these numbers must give the desired value for $\frac{x-z}{y-z}$.
If $x = 10$, $y = 0$ and $z = -1$, then $\frac{x-y}{z-y} = -10$.
In this case, $\frac{x-z}{y-z} = \frac{10-(-1)}{0-(-1)} = \frac{11}{1} = 11$.

Solution 2
Manipulating,

$$\frac{x-z}{y-z} = \frac{(x-y)+(y-z)}{y-z} = \frac{x-y}{y-z} + \frac{y-z}{y-z} = -\frac{x-y}{z-y} + 1$$

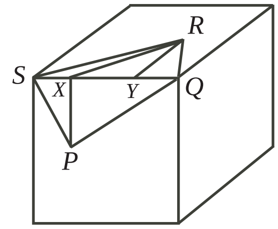
Since $\frac{x-y}{z-y} = -10$, then $\frac{x-z}{y-z} = -(-10) + 1 = 11$.

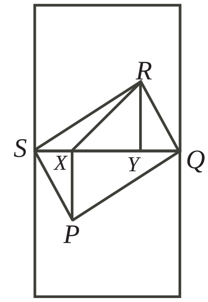
ANSWER: (A)

22. Since $PQRS$ is rectangular, then $\angle SRQ = \angle SPQ = 90^\circ$.
Also, $SR = PQ = 20$ and $SP = QR = 15$.
By the Pythagorean Theorem in $\triangle SPQ$, since $QS > 0$, we have

$$QS = \sqrt{SP^2 + PQ^2} = \sqrt{15^2 + 20^2} = \sqrt{225 + 400} = \sqrt{625} = 25$$

Draw perpendiculars from P and R to X and Y, respectively, on SQ. Also, join R to X.





We want to determine the length of RP.
Now, since $\triangle SPQ$ is right-angled at P, then

$$\sin(\angle PSQ) = \frac{PQ}{SQ} = \frac{20}{25} = \frac{4}{5} \qquad \cos(\angle PSQ) = \frac{SP}{SQ} = \frac{15}{25} = \frac{3}{5}$$

Therefore, $XP = PS\sin(\angle PSQ) = 15(\frac{4}{5}) = 12$ and $SX = PS\cos(\angle PSQ) = 15(\frac{3}{5}) = 9$.
Since $\triangle QRS$ is congruent to $\triangle SPQ$ (three equal side lengths), then $QY = SX = 9$ and $YR = XP = 12$.
Since $SQ = 25$, then $XY = SQ - SX - QY = 25 - 9 - 9 = 7$.
Consider $\triangle RYX$, which is right-angled at Y. By the Pythagorean Theorem,

$$RX^2 = YR^2 + XY^2 = 12^2 + 7^2 = 193$$

Next, consider $\triangle PXR$. Since RX lies in the top face of the cube and PX is perpendicular to this face, then $\triangle PXR$ is right-angled at X.
By the Pythagorean Theorem, since $PR > 0$, we have

$$PR = \sqrt{PX^2 + RX^2} = \sqrt{12^2 + 193} = \sqrt{144 + 193} = \sqrt{337} \approx 18.36$$

Of the given answers, this is closest to 18.4.

ANSWER: (E)

23. First, we try a few values of n to see if we can find a pattern in the values of t_n:

n	$\sqrt{n}$	t_n	n	$\sqrt{n}$	t_n
1	1	1	11	3.32	3
2	1.41	1	12	3.46	3
3	1.73	2	13	3.61	4
4	2	2	14	3.74	4
5	2.24	2	15	3.87	4
6	2.45	2	16	4	4
7	2.65	3	17	4.12	4
8	2.83	3	18	4.24	4
9	3	3	19	4.36	4
10	3.16	3	20	4.47	4
			21	4.58	5

(In each case, the $\sqrt{n}$ column contains an approximation of $\sqrt{n}$ to 2 decimal places.)
So $t_n = 1$ for 2 values of n, 2 for 4 values of n, 3 for 6 values of n, 4 for 8 values of n.
We conjecture that $t_n = k$ for $2k$ values of n. We will prove this fact at the end of the solution.
Next, we note that $\sqrt{2010} \approx 44.83$ and so $t_{2010} = 45$.

This means that before this point in the sequence, we have included all terms with $t_n \leq 44$.
According to our conjecture, the number of terms with $t_n \leq 44$ should be

$$2 + 4 + 6 + \cdots + 86 + 88 = 2(1 + 2 + 3 + \cdots + 43 + 44) = 2\left(\tfrac{1}{2}(44)(45)\right) = 44(45) = 1980$$

Note that $\sqrt{1980} \approx 44.497$ and $\sqrt{1981} \approx 44.508$ so $t_{1980} = 44$ and $t_{1981} = 45$.
Since $t_{1981} = t_{2010} = 45$, then each of the terms from t_{1981} to t_{2010} equals 45. Therefore, there are 30 terms that equal 45.
Thus, the required sum equals

$$2\left(\tfrac{1}{1}\right) + 4\left(\tfrac{1}{2}\right) + 6\left(\tfrac{1}{3}\right) + \cdots + 86\left(\tfrac{1}{43}\right) + 88\left(\tfrac{1}{44}\right) + 30\left(\tfrac{1}{45}\right) = 2 + 2 + 2 + \cdots + 2 + 2 + \tfrac{2}{3}$$

where there are 44 copies of 2.
Therefore, the sum equals $88\frac{2}{3}$.

Lastly, we prove that for each positive integer k, there are $2k$ terms t_n that equal k:

> In order to have $t_n = k$, we need $k - \frac{1}{2} \leq \sqrt{n} < k + \frac{1}{2}$ (in other words, $\sqrt{n}$ needs to round to k).
> Since n and k are positive, then $k - \frac{1}{2} \leq \sqrt{n}$ is equivalent to $(k - \frac{1}{2})^2 \leq n$ and $\sqrt{n} < k + \frac{1}{2}$ is equivalent to $n < (k + \frac{1}{2})^2$.
> Therefore, we need $(k - \frac{1}{2})^2 \leq n < (k + \frac{1}{2})^2$ or $k^2 - k + \frac{1}{4} \leq n < k^2 + k + \frac{1}{4}$.
> Since n is an integer, then $k^2 - k + 1 \leq n \leq k^2 + k$.
> There are thus $(k^2 + k) - (k^2 - k + 1) + 1 = 2k$ such values of n, as required.

ANSWER: (C)

24. First, we fill in the numbers on the top four layers.
The top layer consists of only one sphere, labelled 1.
In the second layer, each sphere touches only one sphere in the layer above. This sphere is labelled 1, so each sphere in the second layer is labelled 1:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}$$

In the third layer, each of the corner spheres touches only one sphere in the second layer and this sphere is labelled 1, so each of the corner spheres on the third layer is labelled 1. The other three spheres (the middle spheres on each edge) touch two spheres each labelled 1 in the layer above, so each is labelled 2. Therefore, the third layer is labelled

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 2 & & 1 \end{array}$$

Similarly, we can complete the fourth layer as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 6 & & 3 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

We define an *external sphere* to be a sphere that is not an internal sphere. In the top four layers, only the sphere labelled 6 in the fourth layer is internal; the remaining spheres are all external. We also use the phrase "the sum of the spheres" to mean the "the sum of the numbers on the spheres".
We observe several patterns:

(i) The corner spheres in each layer are labelled 1.

(ii) The sum of the spheres along an outside edge in the first through fourth layers are 1, 2, 4, 8. It appears that the sum of the spheres along an outside edge in layer k is 2^{k-1}.

(iii) The sums of all of the spheres in the first through fourth layers are 1, 3, 9, 27. It appears that the sum of all of the spheres in layer k is 3^{k-1}.

We use these facts without proof to determine an answer, and then prove these facts.

To determine the sum of the internal spheres, we calculate the sum of all of the spheres and subtract the sum of the external spheres.
Based on fact (iii), the sum of all of the spheres in the 13 layers should be

$$3^0 + 3^1 + 3^2 + \cdots + 3^{11} + 3^{12} = \frac{1(3^{13}-1)}{3-1} = \tfrac{1}{2}(3^{13}-1)$$

(We could calculate the sum of the powers of 3 using a calculator or use the formula for the sum of a geometric series.)
To calculate the sum of all of the external spheres, we consider a fixed layer k, for $k \geq 2$. (The sum of the external spheres in the first layer is 1.)
The external spheres are along the three outside edges, each of which has sum 2^{k-1}, by fact (ii). But using this argument we have included each corner sphere twice (each is included in two edges), so we must subtract 1 for each corner that is doubled. Thus, the sum of the external spheres in layer k should be $3(2^{k-1}) - 3$. (We can check that this formula agrees in each of the first through fourth layers.)
Therefore, the sum of all of the external spheres should be

$$\begin{aligned} 1 + (3(2^1) - 3)) + (3(2^2) - 3) + \cdots + (3(2^{12}) - 3) &= 1 + 3(2^1 + 2^2 + \cdots + 2^{12}) - 36 \\ &= 3\left(\frac{2(2^{12}-1)}{2-1}\right) - 35 \\ &= 3(2^{13} - 2) - 35 \\ &= 3(2^{13}) - 41 \end{aligned}$$

Therefore, the sum of all of the internal spheres should be

$$\tfrac{1}{2}(3^{13}-1) - 3(2^{13}) + 41 = 772\,626$$

Now we must justify the three facts above:

(i) Each corner sphere in layer k touches only one sphere in layer $k-1$, which is itself a corner sphere.
Therefore, the number on a corner sphere in layer k is equal to the number on the corresponding corner sphere in layer $k-1$.
Since the corner spheres are labelled 1 on each of the first four layers, then all corner spheres are labelled 1.

(ii) Consider a fixed edge of spheres in layer k with $k \geq 2$, and consider as well its parallel edge in layer $k+1$.
Consider a sphere, numbered x, on the edge in layer k.
This sphere touches two edge spheres on the parallel edge in layer $k+1$.
Also, spheres from the fixed edge in layer $k+1$ do not touch spheres in layer k that are not on the fixed edge.

The given sphere contributes x to the sum of spheres in the fixed edge in layer k. It thus contributes x to the number on each of the two spheres that it touches in the fixed edge in layer $k+1$.
Therefore, this sphere labelled x in layer k contributes $2x$ to the sum of spheres on the fixed edge in layer $k+1$.
Therefore, the sum of the spheres on the fixed edge in layer $k+1$ is two times the sum of the spheres on the corresponding edge of layer k.
Since the sum of the numbers on the first few layers are powers of 2, then this pattern continues by successively multiplying by 2.

(iii) Suppose a given sphere in layer k is labelled x.
This sphere touches three spheres in layer $k+1$.
Therefore, the sphere contributes x to the sum in layer k, and $3x$ to the sum in layer $k+1$ (x to each of 3 spheres).
Therefore, the sum of the spheres in layer $k+1$ is three times the sum of the spheres in layer k, since each sphere from layer k contributes three times in layer $k+1$.
Since the sum of the numbers on the first few layers are powers of 3, then this pattern continues by successively multiplying by 3.

Answer: (E)

25. Define $f(x) = (1-x)^a(1+x)^b(1-x+x^2)^c(1+x^2)^d(1+x+x^2)^e(1+x+x^2+x^3+x^4)^f$.
We note several algebraic identities, each of which can be checked by expanding and simplifying:

$$\begin{aligned}
1-x^5 &= (1-x)(1+x+x^2+x^3+x^4)\\
1-x^3 &= (1-x)(1+x+x^2)\\
1-x^4 &= (1-x^2)(1+x^2) = (1-x)(1+x)(1+x^2)\\
1+x^3 &= (1+x)(1-x+x^2)\\
1-x^6 &= (1-x^3)(1+x^3) = (1-x)(1+x)(1-x+x^2)(1+x+x^2)
\end{aligned}$$

This allows us to regroup the terms successively in the given expansion to create the simpler left sides in the equation above:

$$\begin{aligned}
&(1-x)^a(1+x)^b(1-x+x^2)^c(1+x^2)^d(1+x+x^2)^e(1+x+x^2+x^3+x^4)^f\\
&= (1-x)^{a-e-f}(1+x)^b(1-x+x^2)^c(1+x^2)^d(1+x+x^2)^e(1-x)^e(1+x+x^2+x^3+x^4)^f(1-x)^f\\
&= (1-x)^{a-e-f}(1+x)^b(1-x+x^2)^c(1+x^2)^d(1-x^3)^e(1-x^5)^f\\
&= (1-x)^{a-e-f}(1+x)^{b-c}(1-x+x^2)^c(1+x)^c(1+x^2)^d(1-x^3)^e(1-x^5)^f\\
&= (1-x)^{a-e-f}(1+x)^{b-c}(1+x^3)^c(1+x^2)^d(1-x^3)^e(1-x^5)^f\\
&= (1-x)^{a-e-f}(1+x)^{b-c}(1+x^3)^c(1-x^3)^c(1+x^2)^d(1-x^3)^{e-c}(1-x^5)^f\\
&= (1-x)^{a-e-f}(1+x)^{b-c}(1-x^6)^c(1+x^2)^d(1-x^3)^{e-c}(1-x^5)^f\\
&= (1-x)^{a-e-f-d}(1+x)^{b-c-d}(1-x^6)^c(1+x^2)^d(1-x)^d(1+x)^d(1-x^3)^{e-c}(1-x^5)^f\\
&= (1-x)^{a-e-f-d}(1+x)^{b-c-d}(1-x^6)^c(1-x^4)^d(1-x^3)^{e-c}(1-x^5)^f\\
&= (1-x)^{a-d-e-f}(1+x)^{b-c-d}(1-x^3)^{e-c}(1-x^4)^d(1-x^5)^f(1-x^6)^c
\end{aligned}$$

Since $a > d+e+f$ and $e > c$ and $b > c+d$, then the exponents $a-d-e-f$ and $b-c-d$ and $e-c$ are positive integers.
Define $A = a-d-e-f$, $B = b-c-d$, $C = e-c$, $D = d$, $E = f$, and $F = c$.
We want the expansion of

$$f(x) = (1-x)^A(1+x)^B(1-x^3)^C(1-x^4)^D(1-x^5)^E(1-x^6)^F$$

to have only terms $1 - 2x$ when all terms involving x^7 or larger are removed.
We use the facts that

$$(1 + y)^n = 1 + ny + \frac{n(n-1)}{2}y^2 + \cdots \qquad (*)$$

and

$$(1 - y)^n = 1 - ny + \frac{n(n-1)}{2}y^2 + \cdots \qquad (**)$$

which can be derived by multiplying out directly or by using the Binomial Theorem.

Since each factor contains a constant term of 1, then $f(x)$ will have a constant term of 1, regardless of the values of A, B, C, D, E, F.
Consider the first two factors. Only these factors can affect the coefficients of x and x^2 in the final product. Any terms involving x and x^2 in the final product will come from these factors multiplied by the constant 1 from each of the other factors.
We consider the product of the first two factors ignoring any terms of degree three or higher:

$$\begin{aligned}(1-x)^A(1+x)^B &= (1 - Ax + \frac{A(A-1)}{2}x^2 - \cdots)(1 + Bx + \frac{B(B-1)}{2}x^2 + \cdots) \\ &= 1 - Ax + \frac{A(A-1)}{2}x^2 + Bx - ABx^2 + \frac{B(B-1)}{2}x^2 + \cdots \\ &= 1 - (A-B)x + \left[\frac{A(A-1)}{2} + \frac{B(B-1)}{2} - AB\right]x^2 + \cdots\end{aligned}$$

These will be the terms involving 1, x and x^2 in the final expansion of $f(x)$.
Since $f(x)$ has a term $-2x$ and no x^2 term, then $A - B = 2$ and $\frac{A(A-1)}{2} + \frac{B(B-1)}{2} - AB = 0$.
The second equation becomes $A^2 - A + B^2 - B - 2AB = 0$ or $(A - B)^2 = A + B$.
Since $A - B = 2$, then $A + B = 4$, whence $2A = (A + B) + (A - B) = 6$, so $A = 3$ and $B = 1$.
Thus, the first two factors are $(1 - x)^3(1 + x)$.
Note that $(1 - x)^3(1 + x) = (1 - 3x + 3x^2 - x^3)(1 + x) = 1 - 2x + 2x^3 - x^4$.
Therefore, $f(x) = (1 - 2x + 2x^3 - x^4)(1 - x^3)^C(1 - x^4)^D(1 - x^5)^E(1 - x^6)^F$.
The final result contains no x^3 term. Since the first factor contains a term "$+2x^3$" which will appear in the final product by multiplying by all of the constant terms in subsequent factors, then this "$+2x^3$" must be balanced by a "$-2x^3$". The only other factor possibly containing an x^3 is $(1 - x^3)^C$. To balance the "$+2x^3$" term, the expansion of $(1 - x^3)^C$ must include a term "$-2x^3$" which will be multiplied by the constant terms in the other factors to provide a "$-2x^3$" in the final expansion, balancing the "$+2x^3$".
For $(1 - x^3)^C$ to include $-2x^3$, we must have $C = 2$, from $(*)$.
Therefore,

$$\begin{aligned}f(x) &= (1 - 2x + 2x^3 - x^4)(1 - x^3)^2(1 - x^4)^D(1 - x^5)^E(1 - x^6)^F \\ &= (1 - 2x + 2x^3 - x^4)(1 - 2x^3 + x^6)(1 - x^4)^D(1 - x^5)^E(1 - x^6)^F \\ &= (1 - 2x + 3x^4 - 3x^6 + \cdots)(1 - x^4)^D(1 - x^5)^E(1 - x^6)^F\end{aligned}$$

When we simplify at this stage, we can ignore any terms with exponent greater than 6, since we do not care about these terms and they do not affect terms with smaller exponents when we multiply out.
To balance the "$+3x^4$", the factor $(1 - x^4)^D$ needs to include "$-3x^4$" and so $D = 3$.

Therefore,

$$\begin{aligned} f(x) &= (1-2x+3x^4-3x^6+\cdots)(1-x^4)^3(1-x^5)^E(1-x^6)^F \\ &= (1-2x+3x^4-3x^6+\cdots)(1-3x^4+\cdots)(1-x^5)^E(1-x^6)^F \\ &= (1-2x+6x^5-3x^6+\cdots)(1-x^5)^E(1-x^6)^F \end{aligned}$$

To balance the "$+6x^5$", the factor $(1-x^5)^E$ needs to include "$-6x^5$" and so $E=6$.
Therefore,

$$\begin{aligned} f(x) &= (1-2x+6x^5-3x^6+\cdots)(1-x^5)^6(1-x^6)^F \\ &= (1-2x+6x^5-3x^6+\cdots)(1-6x^5+\cdots)(1-x^6)^F \\ &= (1-2x+9x^6+\cdots)(1-x^6)^F \end{aligned}$$

To balance the "$+9x^6$", the factor $(1-x^6)^F$ needs to include "$-9x^6$" and so $F=9$.

We now know that $A=3$, $B=1$, $C=2$, $D=3$, $E=6$, and $F=9$.
Since $D=d$, $E=f$, and $F=c$, then $c=9$, $f=6$, and $d=3$.
Since $C=e-c$, $C=2$ and $c=9$, then $e=11$.
Since $A=a-d-e-f$, $d=3$, $e=11$, $f=6$, and $A=3$, then $a=3+3+11+6$, or $a=23$.

Answer: (E)