



The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

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Euclid Contest

Tuesday, April 12, 2011

UNIVERSITY OF
WATERLOO

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MATHEMATICS

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Time: $2\frac{1}{2}$ hours

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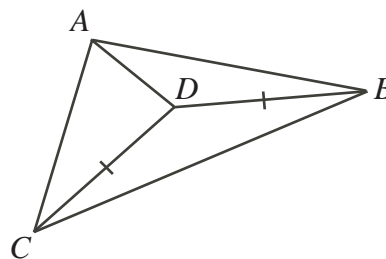
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1.  (a) If $(x + 1) + (x + 2) + (x + 3) = 8 + 9 + 10$, what is the value of x ?
 (b) If $\sqrt{25} + \sqrt{x} = 6$, what is the value of x ?
 (c) The point $(a, 2)$ is the point of intersection of the lines with equations $y = 2x - 4$ and $y = x + k$. Determine the value of k .

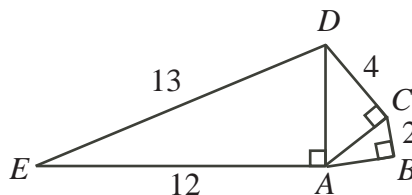
2.  (a) An equilateral triangle of side length 1 is cut out of the middle of each side of a square of side length 3, as shown. What is the perimeter of the resulting figure?




-  (b) In the diagram, $DC = DB$, $\angle DCB = 15^\circ$, and $\angle ADB = 130^\circ$. What is the measure of $\angle ADC$?



-  (c) In the diagram, $\angle EAD = 90^\circ$, $\angle ACD = 90^\circ$, and $\angle ABC = 90^\circ$. Also, $ED = 13$, $EA = 12$, $DC = 4$, and $CB = 2$. Determine the length of AB .



3.  (a) If $2 \leq x \leq 5$ and $10 \leq y \leq 20$, what is the maximum value of $15 - \frac{y}{x}$?




- (b) The functions f and g satisfy

$$f(x) + g(x) = 3x + 5$$

$$f(x) - g(x) = 5x + 7$$

for all values of x . Determine the value of $2f(2)g(2)$.

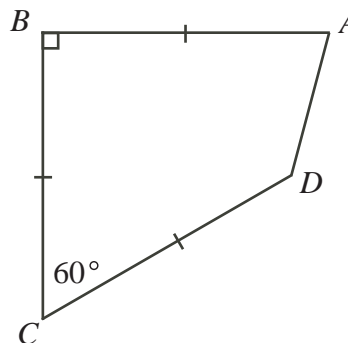
4.  (a) Three different numbers are chosen at random from the set $\{1, 2, 3, 4, 5\}$. The numbers are arranged in increasing order.


What is the probability that the resulting sequence is an arithmetic sequence?

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9 is an arithmetic sequence with four terms.)



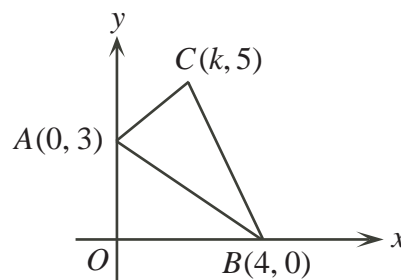
- (b) In the diagram, $ABCD$ is a quadrilateral with $AB = BC = CD = 6$, $\angle ABC = 90^\circ$, and $\angle BCD = 60^\circ$. Determine the length of AD .




5.  (a) What is the largest two-digit number that becomes 75% greater when its digits are reversed?




- (b) A triangle has vertices $A(0, 3)$, $B(4, 0)$, $C(k, 5)$, where $0 < k < 4$. If the area of the triangle is 8, determine the value of k .



6.  (a) Serge likes to paddle his raft down the Speed River from point A to point B . The speed of the current in the river is always the same. When Serge paddles, he always paddles at the same constant speed. On days when he paddles with the current, it takes him 18 minutes to get from A to B . When he does not paddle, the current carries him from A to B in 30 minutes. If there were no current, how long would it take him to paddle from A to B ?




- (b) Square $OPQR$ has vertices $O(0, 0)$, $P(0, 8)$, $Q(8, 8)$, and $R(8, 0)$. The parabola with equation $y = a(x - 2)(x - 6)$ intersects the sides of the square $OPQR$ at points K , L , M , and N . Determine all the values of a for which the area of the trapezoid $KLMN$ is 36.

7.  (a) A 75 year old person has a 50% chance of living at least another 10 years.
 A 75 year old person has a 20% chance of living at least another 15 years.
 An 80 year old person has a 25% chance of living at least another 10 years.
 What is the probability that an 80 year old person will live at least another 5 years?





- (b) Determine all values of x for which $2^{\log_{10}(x^2)} = 3(2^{1+\log_{10} x}) + 16$.

8.  The Sieve of Sundaram uses the following infinite table of positive integers:

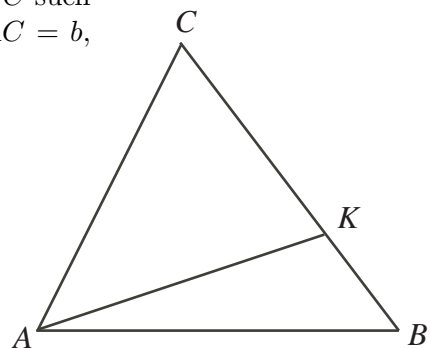
4	7	10	13	...
7	12	17	22	...
10	17	24	31	...
13	22	31	40	...
\vdots	\vdots	\vdots	\vdots	

The numbers in each row in the table form an arithmetic sequence. The numbers in each column in the table form an arithmetic sequence. The first four entries in each of the first four rows and columns are shown.

- (a) Determine the number in the 50th row and 40th column.
 (b) Determine a formula for the number in the R th row and C th column.
 (c) Prove that if N is an entry in the table, then $2N + 1$ is composite.
9.  Let $[x]$ denote the greatest integer less than or equal to x . For example, $[3.1] = 3$ and $[-1.4] = -2$.
- Suppose that $f(n) = 2n - \left\lfloor \frac{1 + \sqrt{8n - 7}}{2} \right\rfloor$ and $g(n) = 2n + \left\lfloor \frac{1 + \sqrt{8n - 7}}{2} \right\rfloor$ for each positive integer n .
- (a) Determine the value of $g(2011)$.
 (b) Determine a value of n for which $f(n) = 100$.
 (c) Suppose that $A = \{f(1), f(2), f(3), \dots\}$ and $B = \{g(1), g(2), g(3), \dots\}$; that is, A is the range of f and B is the range of g . Prove that every positive integer m is an element of exactly one of A or B .

10.  In the diagram, $2\angle BAC = 3\angle ABC$ and K lies on BC such that $\angle KAC = 2\angle KAB$. Suppose that $BC = a$, $AC = b$, $AB = c$, $AK = d$, and $BK = x$.

- (a) Prove that $d = \frac{bc}{a}$ and $x = \frac{a^2 - b^2}{a}$.
 (b) Prove that $(a^2 - b^2)(a^2 - b^2 + ac) = b^2c^2$.
 (c) Determine a triangle with positive integer side lengths a, b, c and positive area that satisfies the condition in part (b).





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Canadian Mathematics Competition

An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

Euclid Contest

Wednesday, April 7, 2010



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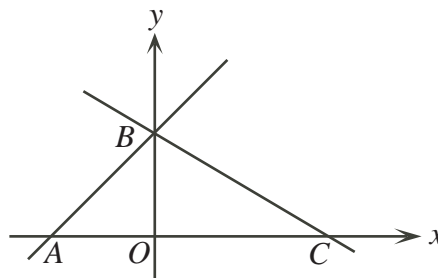
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1.  (a) If $3^x = 27$, what is the value of 3^{x+2} ?
-  (b) If $2^5 3^{13} 5^9 x = 2^7 3^{14} 5^9$, what is the value of x ?

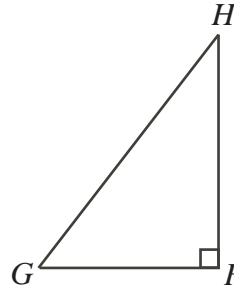



- (c) Triangle ABC is enclosed by the lines $y = x + 2$, $y = -\frac{1}{2}x + 2$ and the x -axis. Determine the area of $\triangle ABC$.

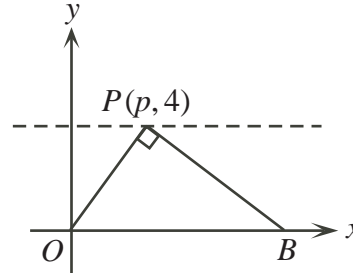



2.  (a) Maria has a red package, a green package, and a blue package. The sum of the masses of the three packages is 60 kg. The sum of the masses of the red and green packages is 25 kg. The sum of the masses of the green and blue packages is 50 kg. What is the mass of the green package, in kg?
-  (b) A *palindrome* is a positive integer that is the same when read forwards or backwards. For example, 151 is a palindrome. What is the largest palindrome less than 200 that is the sum of three consecutive integers?
-  (c) If $(x + 1)(x - 1) = 8$, determine the numerical value of $(x^2 + x)(x^2 - x)$.


3.  (a) Bea the bee sets out from her hive, H , and flies south for 1 hour to a field, F . She spends 30 minutes in the field, and then flies 45 minutes west to a garden, G . After spending 1 hour in the garden, she flies back to her hive along a straight line route. Bea always flies at the same constant speed. What is the total length of time, in minutes, that she is away from her hive?





-  (b) In the diagram, points $P(p, 4)$, $B(10, 0)$, and $O(0, 0)$ are shown. If $\triangle OPB$ is right-angled at P , determine all possible values of p .




4.  (a) Thurka bought some stuffed goats and some toy helicopters. She paid a total of \$201. She did not buy partial goats or partial helicopters. Each stuffed goat cost \$19 and each toy helicopter cost \$17. How many of each did she buy?

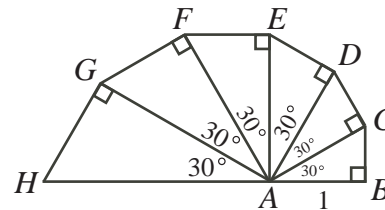
-  (b) Determine all real values of x for which $(x + 8)^4 = (2x + 16)^2$.


5.  (a) If $f(x) = 2x + 1$ and $g(f(x)) = 4x^2 + 1$, determine an expression for $g(x)$.

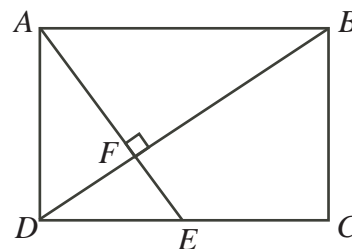
-  (b) A geometric sequence has 20 terms.
The sum of its first two terms is 40.
The sum of its first three terms is 76.
The sum of its first four terms is 130.
Determine how many of the terms in the sequence are integers.

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)


6.  (a) A snail's shell is formed from six triangular sections, as shown. Each triangle has interior angles of 30° , 60° and 90° . If AB has a length of 1 cm, what is the length of AH , in cm?





-  (b) In rectangle $ABCD$, point E is on side DC . Line segments AE and BD are perpendicular and intersect at F . If $AF = 4$ and $DF = 2$, determine the area of quadrilateral $BCEF$.

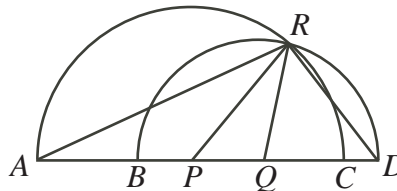



7.  (a) Determine all real values of x for which $3^{(x-1)}9^{\frac{3}{2x^2}} = 27$.

 (b) Determine all points (x, y) where the two curves $y = \log_{10}(x^4)$ and $y = (\log_{10} x)^3$ intersect.

8.  (a) Oi-Lam tosses three fair coins and removes all of the coins that come up heads. George then tosses the coins that remain, if any. Determine the probability that George tosses exactly one head.

 (b) In the diagram, points $B, P, Q,$ and C lie on line segment AD . The semi-circle with diameter AC has centre P and the semi-circle with diameter BD has centre Q . The two semi-circles intersect at R . If $\angle PRQ = 40^\circ$, determine the measure of $\angle ARD$.




9.  (a) (i) If θ is an angle whose measure is not an integer multiple of 90° , prove that


$$\cot \theta - \cot 2\theta = \frac{1}{\sin 2\theta}$$

(ii) Ross starts with an angle of measure 8° and doubles it 10 times until he obtains 8192° . He then adds up the reciprocals of the sines of these 11 angles. That is, he calculates

$$S = \frac{1}{\sin 8^\circ} + \frac{1}{\sin 16^\circ} + \frac{1}{\sin 32^\circ} + \cdots + \frac{1}{\sin 4096^\circ} + \frac{1}{\sin 8192^\circ}$$

Determine, without using a calculator, the measure of the acute angle α so that $S = \frac{1}{\sin \alpha}$.

 (b) In $\triangle ABC$, $BC = a$, $AC = b$, $AB = c$, and $a < \frac{1}{2}(b + c)$. Prove that $\angle BAC < \frac{1}{2}(\angle ABC + \angle ACB)$.

10.  For each positive integer n , let $T(n)$ be the number of triangles with integer side lengths, positive area, and perimeter n . For example, $T(6) = 1$ since the only such triangle with a perimeter of 6 has side lengths 2, 2 and 2.

(a) Determine the values of $T(10)$, $T(11)$ and $T(12)$.

(b) If m is a positive integer with $m \geq 3$, prove that $T(2m) = T(2m - 3)$.

(c) Determine the smallest positive integer n such that $T(n) > 2010$.



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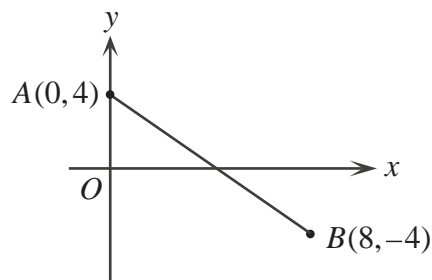
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1.  (a) A line has equation $6x + 3y - 21 = 0$. What is the slope of the line?


 (b) A line with a slope of 3 passes through the points $(1, 0)$ and $(5, c)$. What is the value of c ?


 (c) The point (k, k) lies on the line segment AB shown in the diagram. Determine the value of k .

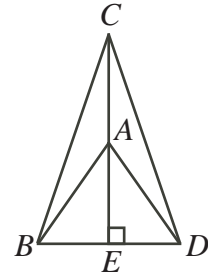



2.  (a) What is the sum of the two numbers that satisfy the equation $x^2 - 6x - 7 = 0$?

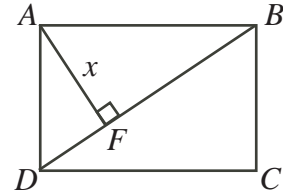
 (b) What is the product of the two numbers that satisfy the equation $5x^2 - 20 = 0$?


 (c) Determine the average of the numbers that satisfy the equation $x^3 - 6x^2 + 5x = 0$.


3.  (a) In the diagram, $AB = AC = AD = BD$ and CAE is a straight line segment that is perpendicular to BD . What is the measure of $\angle CDB$?




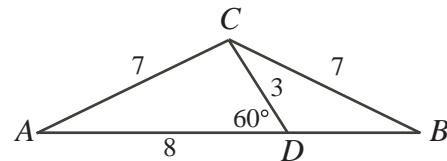
-  (b) In rectangle $ABCD$, F is on diagonal BD so that AF is perpendicular to BD . Also, $BC = 30$, $CD = 40$ and $AF = x$. Determine the value of x .




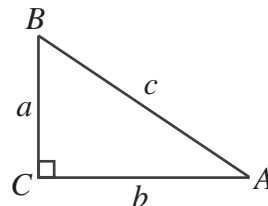
4.  (a) In an arithmetic sequence, the first term is 1 and the last term is 19. The sum of all the terms in the sequence is 70. How many terms does the sequence have?
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9 is an arithmetic sequence with four terms.)


-  (b) Suppose that $a(x + b(x + 3)) = 2(x + 6)$ for *all* values of x . Determine a and b .


5.  (a) In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60^\circ$, $AD = 8$, and $CD = 3$. Determine the length of BD .

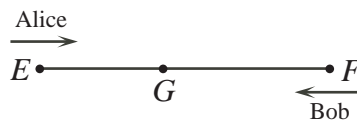


-  (b) In the diagram, $\triangle ABC$ is right-angled at C . Also, $2 \sin B = 3 \tan A$. Determine the measure of angle A .




6.  (a) An integer n , with $100 \leq n \leq 999$, is chosen at random. What is the probability that the sum of the digits of n is 24?

-  (b) Alice drove from town E to town F at a constant speed of 60 km/h. Bob drove from F to E along the same road also at a constant speed. They started their journeys at the same time and passed each other at point G .

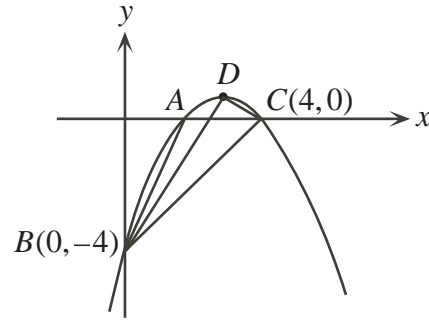


Alice drove from G to F in 45 minutes. Bob drove from G to E in 20 minutes. Determine Bob's constant speed.

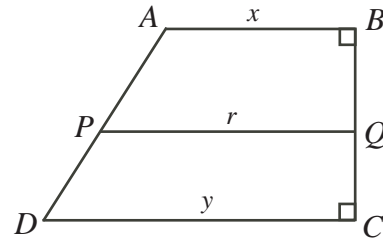
7.  (a) The parabola $y = x^2 - 2x + 4$ is translated p units to the right and q units down. The x -intercepts of the resulting parabola are 3 and 5. What are the values of p and q ?



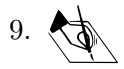
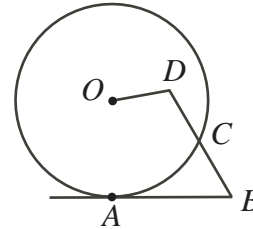
- (b) In the diagram, D is the vertex of a parabola. The parabola cuts the x -axis at A and at $C(4, 0)$. The parabola cuts the y -axis at $B(0, -4)$. The area of $\triangle ABC$ is 4. Determine the area of $\triangle DBC$.



- (a) $ABCD$ is a trapezoid with parallel sides AB and DC . Also, BC is perpendicular to AB and to DC . The line PQ is parallel to AB and to DC . The line PQ divides the trapezoid into two regions of equal area. If $AB = x$, $DC = y$, and $PQ = r$, prove that $x^2 + y^2 = 2r^2$.



- (b) In the diagram, AB is tangent to the circle with centre O and radius r . The length of AB is p . Point C is on the circle and D is inside the circle so that BCD is a straight line, as shown. If $BC = CD = DO = q$, prove that $q^2 + r^2 = p^2$.

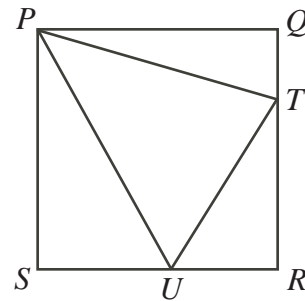


- (a) If $\log_2 x$, $(1 + \log_4 x)$, and $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible values of x .

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

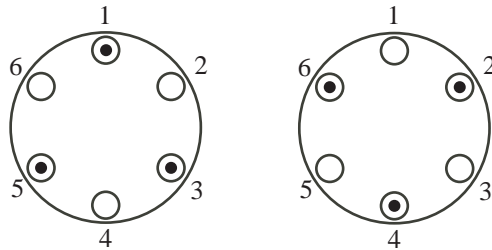


- (b) In the diagram, $PQRS$ is a square with sides of length 4. Points T and U are on sides QR and RS respectively such that $\angle UPT = 45^\circ$. Determine the maximum possible perimeter of $\triangle RUT$.





10. Suppose there are n plates equally spaced around a circular table. Ross wishes to place an identical gift on each of k plates, so that no two neighbouring plates have gifts. Let $f(n, k)$ represent the number of ways in which he can place the gifts. For example $f(6, 3) = 2$, as shown below.



- (a) Determine the value of $f(7, 3)$.
- (b) Prove that $f(n, k) = f(n - 1, k) + f(n - 2, k - 1)$ for all integers $n \geq 3$ and $k \geq 2$.
- (c) Determine the smallest possible value of $n + k$ among all possible ordered pairs of integers (n, k) for which $f(n, k)$ is a positive multiple of 2009, where $n \geq 3$ and $k \geq 2$.



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Canadian Mathematics Competition

An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

Euclid Contest

Tuesday, April 15, 2008

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Time: 2½ hours

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2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.


NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

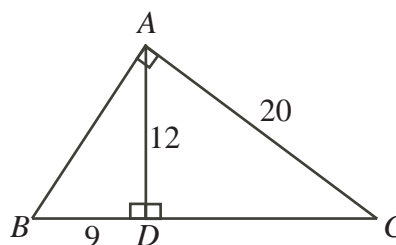
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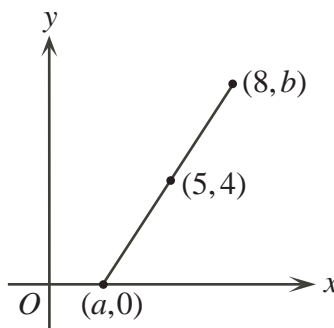
A Note about Writing Solutions


For each problem marked “  ”, a full solution is required. The solutions that you provide in the answer booklet should be well organized and contain a balance of mathematical statements and words of explanation. Working out some of the details in rough on a separate piece of paper before writing your finished solution is a good idea. Your final solution should be written so that the marker can understand your approach to the problem and all of the mathematical steps of your solution.

1.  (a) In the diagram, what is the perimeter of $\triangle ABC$?




-  (b) In the diagram, the line segment with endpoints $(a, 0)$ and $(8, b)$ has midpoint $(5, 4)$. What are the values of a and b ?





-  (c) The lines $ax + y = 30$ and $x + ay = k$ intersect at the point $P(6, 12)$. Determine the value of k .

2. Each part of this problem refers to the parabola $y = (x - 2)(x - 8) + 7$.

-  (a) The points $(2, 7)$ and $(c, 7)$, where $c \neq 2$, lie on the parabola. What is the value of c ?

-  (b) What are the coordinates of the vertex of the parabola?

-  (c) A line that passes through the point $A(5, 0)$ intersects the parabola at $B(4, -1)$. Determine the other point at which this line intersects the parabola.

3.  (a) A 3×3 square frame is placed on a grid of numbers, as shown. In the example, the sum of the numbers inside the square frame is 108, and the middle number is 12. When the square frame is moved to a new position, the sum of its numbers becomes 279. In the frame's new position, what is the middle number?

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49



- (b) Of the three figures shown, which has the smallest area and which has the largest area? Explain how you determined your answer. (In Figure A, the circle has a diameter of length 2.)

Figure A

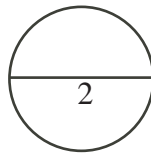


Figure B

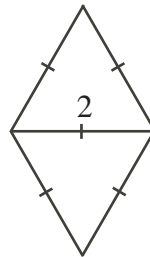
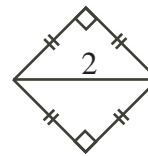

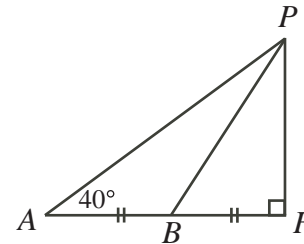


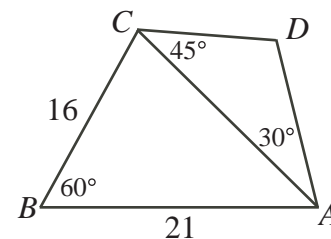
Figure C




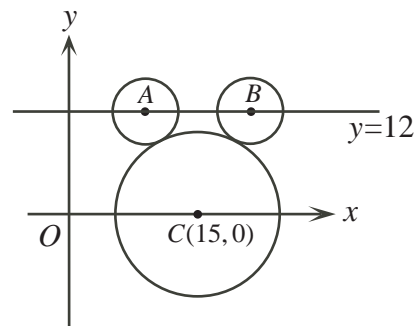
4.  (a) A flagpole FP is 20 metres tall. From point A on the flat ground, the angle of elevation to the top of the flagpole is 40° . If B is halfway from A to F , what is the measure of $\angle FBP$, to the nearest degree?



- (b) In the diagram, $AB = 21$ and $BC = 16$. Also, $\angle ABC = 60^\circ$, $\angle CAD = 30^\circ$, and $\angle ACD = 45^\circ$. Determine the length of CD , to the nearest tenth.



5.  (a) In the diagram, the large circle has radius 9 and centre $C(15,0)$. The small circles have radius 4 and centres A and B on the horizontal line $y = 12$. Each of the two small circles is tangent to the large circle. It takes a bug 5 seconds to walk at a constant speed from A to B along the line $y = 12$. How far does the bug walk in 1 second?





- (b) Determine all values of k , with $k \neq 0$, for which the parabola

$$y = kx^2 + (5k + 3)x + (6k + 5)$$

has its vertex on the x -axis.

6.



- (a) The function $f(x)$ satisfies the equation $f(x) = f(x - 1) + f(x + 1)$ for all values of x . If $f(1) = 1$ and $f(2) = 3$, what is the value of $f(2008)$?



- (b) The numbers a, b, c , in that order, form a three term arithmetic sequence (see below) and $a + b + c = 60$.
The numbers $a - 2, b, c + 3$, in that order, form a three term geometric sequence. Determine all possible values of a, b and c .

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7 is an arithmetic sequence with three terms.

A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

7.



- (a) The average of three consecutive multiples of 3 is a .
The average of four consecutive multiples of 4 is $a + 27$.
The average of the smallest and largest of these seven integers is 42.
Determine the value of a .

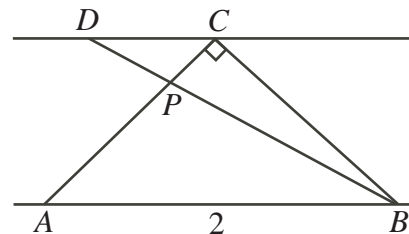


- (b) Billy and Crystal each have a bag of 9 balls. The balls in each bag are numbered from 1 to 9. Billy and Crystal each remove one ball from their own bag. Let b be the sum of the numbers on the balls remaining in Billy's bag. Let c be the sum of the numbers on the balls remaining in Crystal's bag. Determine the probability that b and c differ by a multiple of 4.

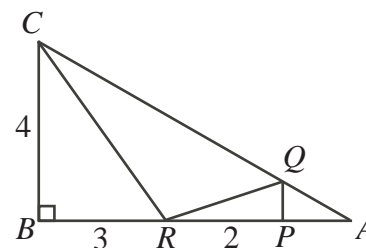
8.




- (a) Points A, B, C , and D are arranged, as shown, with AB parallel to DC and P the point of intersection of AC and BD . Also, $\angle ACB = 90^\circ$, $AC = CB$, $AB = BD = 2$. Determine the measure of $\angle DBC$.



- (b) In the diagram, ABC is a right-angled triangle with P and R on AB . Also, Q is on AC , and PQ is parallel to BC . If $RP = 2$, $BR = 3$, $BC = 4$, and the area of $\triangle QRC$ is 5, determine the length of AP .




9.  (a) The equation $2^{x+2}5^{6-x} = 10^{x^2}$ has two real solutions. Determine these two solutions.



- (b) Determine all real solutions to the system of equations

$$\begin{aligned}x + \log_{10} x &= y - 1 \\y + \log_{10}(y - 1) &= z - 1 \\z + \log_{10}(z - 2) &= x + 2\end{aligned}$$

and prove that there are no more solutions.

10.  Suppose that n is a positive integer. Consider an upward-pointing equilateral triangle of side length n , cut up into unit triangles, as shown.

$$n = 3$$



$$n = 4$$



For each n , let $f(n)$ represent the total number of downward-pointing equilateral triangles of *all* sizes. For example, $f(3) = 3$ and $f(4) = 6 + 1 = 7$, as illustrated below.

$$n = 3$$



$$f(3) = 3$$

$$n = 4$$



$$f(4) = 6 + 1 = 7$$

- (a) Determine the values of $f(5)$ and $f(6)$.
 (b) Prove that $f(2k) = f(2k - 1) + k^2$ for each positive integer $k \geq 1$.
 (c) Determine, with justification, all positive integers n for which $f(n)$ is divisible by n .



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Canadian Mathematics Competition

An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

Euclid Contest

Tuesday, April 17, 2007

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Time: $2\frac{1}{2}$ hours

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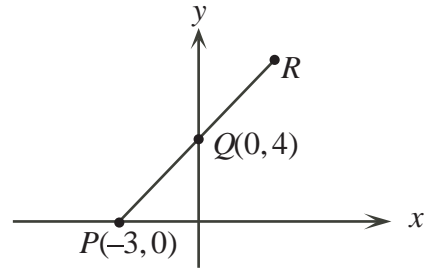
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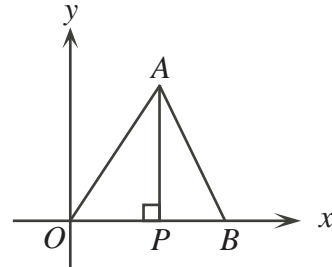
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1.  (a) If the point $(a - 1, a + 1)$ lies on the line $y = 2x - 3$, what is the value of a ?

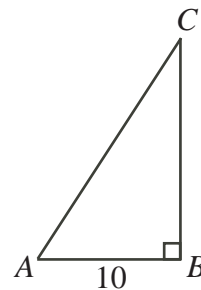
 (b) In the diagram, a line is drawn through points P , Q and R . If $PQ = QR$, what are the coordinates of R ?





 (c) In the diagram, $OA = 15$, $OP = 9$ and $PB = 4$. Determine the equation of the line through A and B . Explain how you got your answer.

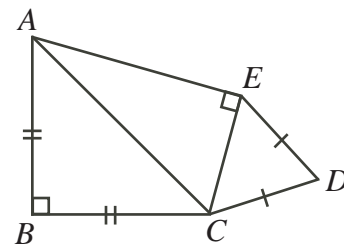



2.  (a) In the diagram, $\triangle ABC$ is right-angled at B and $AB = 10$. If $\cos(\angle BAC) = \frac{5}{13}$, what is the value of $\tan(\angle ACB)$?



 (b) Suppose $0^\circ < x < 90^\circ$ and $2\sin^2 x + \cos^2 x = \frac{25}{16}$. What is the value of $\sin x$?


 (c) In the diagram, $AB = BC = 2\sqrt{2}$, $CD = DE$, $\angle CDE = 60^\circ$, and $\angle EAB = 75^\circ$. Determine the perimeter of figure $ABCDE$. Explain how you got your answer.



3.  (a) The first term of a sequence is 2007. Each term, starting with the second, is the sum of the cubes of the digits of the previous term. What is the 2007th term?

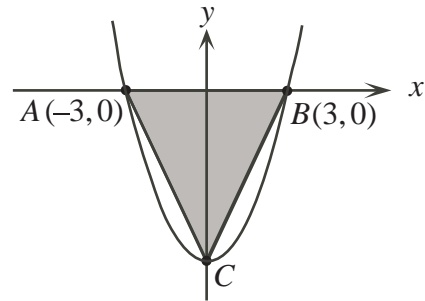



- (b) Sequence A has n th term $n^2 - 10n + 70$.
 (The first three terms of sequence A are 61, 54, 49.)
 Sequence B is an arithmetic sequence with first term 5 and common difference 10.
 (The first three terms of sequence B are 5, 15, 25.)
 Determine all n for which the n th term of sequence A is equal to the n th term of sequence B. Explain how you got your answer.

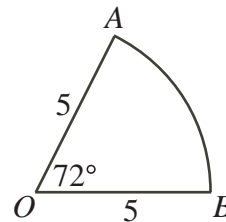
4.  (a) Determine all values of x for which $2 + \sqrt{x-2} = x - 2$.



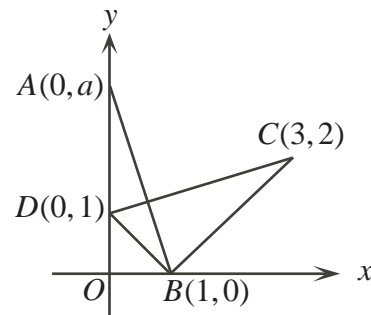
- (b) In the diagram, the parabola intersects the x -axis at $A(-3, 0)$ and $B(3, 0)$ and has its vertex at C below the x -axis. The area of $\triangle ABC$ is 54. Determine the equation of the parabola. Explain how you got your answer.




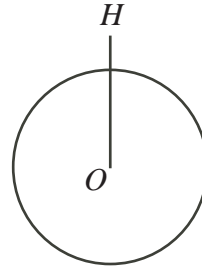
5.  (a) In the diagram, a sector of a circle with centre O and radius 5 is shown. What is the perimeter of the sector?




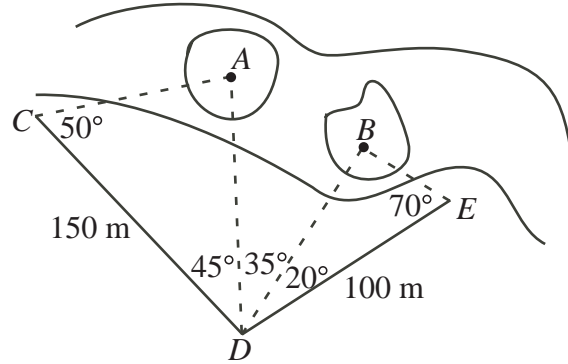
- (b) In the diagram, $A(0, a)$ lies on the y -axis above D . If the triangles AOB and BCD have the same area, determine the value of a . Explain how you got your answer.





6.  (a) The Little Prince lives on a spherical planet which has a radius of 24 km and centre O . He hovers in a helicopter (H) at a height of 2 km above the surface of the planet. From his position in the helicopter, what is the distance, in kilometres, to the furthest point on the surface of the planet that he can see?

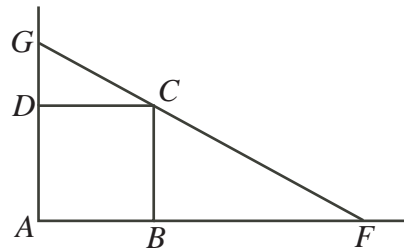



-  (b) In the diagram, points A and B are located on islands in a river full of rabid aquatic goats. Determine the distance from A to B , to the nearest metre. (Luckily, someone has measured the angles shown in the diagram as well as the distances CD and DE .)

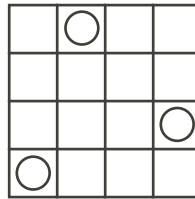



7.  (a) Determine all values of x for which $(\sqrt{x})^{\log_{10} x} = 100$.

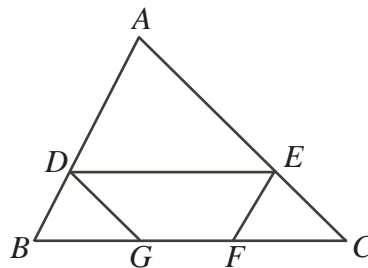
-  (b) In the diagram, line segment FCG passes through vertex C of square $ABCD$, with F lying on AB extended and G lying on AD extended. Prove that $\frac{1}{AB} = \frac{1}{AF} + \frac{1}{AG}$.




8.  (a) In the 4×4 grid shown, three coins are randomly placed in different squares. Determine the probability that no two coins lie in the same row or column.




-  (b) In the diagram, the area of $\triangle ABC$ is 1. Trapezoid $DEFG$ is constructed so that G is to the left of F , DE is parallel to BC , EF is parallel to AB and DG is parallel to AC . Determine the maximum possible area of trapezoid $DEFG$.

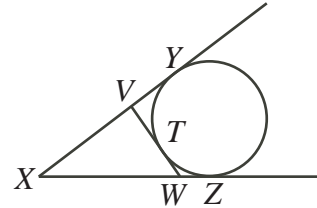


9.  The parabola $y = f(x) = x^2 + bx + c$ has vertex P and the parabola $y = g(x) = -x^2 + dx + e$ has vertex Q , where P and Q are distinct points. The two parabolas also intersect at P and Q .

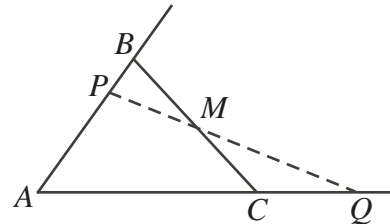
(a) Prove that $2(e - c) = bd$.

(b) Prove that the line through P and Q has slope $\frac{1}{2}(b + d)$ and y -intercept $\frac{1}{2}(c + e)$.

10.  (a) In the diagram, the circle is tangent to XY at Y and to XZ at Z . Point T is chosen on the minor arc YZ and a tangent to the circle is drawn at T , cutting XY at V and XZ at W . Prove that the perimeter of $\triangle V X W$ is independent of the position of T .



- (b) In the diagram, $AB = 10$, $BC = 14$, $AC = 16$, and M is the midpoint of BC . Various lines can be drawn through M , cutting AB (possibly extended) at P and AC (possibly extended) at Q . Determine, with proof, the minimum possible perimeter of $\triangle APQ$.





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Canadian Mathematics Competition

An activity of the Centre for Education
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University of Waterloo, Waterloo, Ontario

Euclid Contest

Wednesday, April 19, 2006

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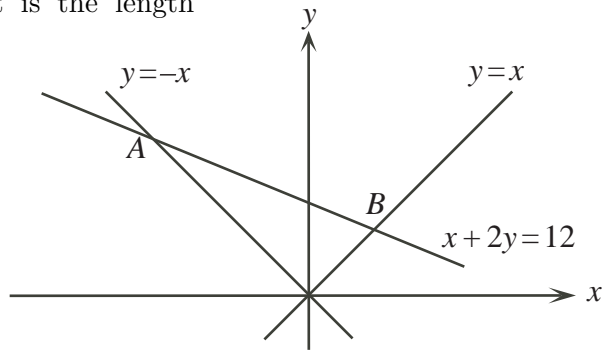
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-  (a) What is the sum of the x -intercept and the y -intercept of the line $3x - 3y = 24$?
  (b) If the lines $px = 12$ and $2x + qy = 10$ intersect at $(1, 1)$, what is the value of $p + q$?
  (c) In the diagram, the line $x + 2y = 12$ intersects the lines $y = -x$ and $y = x$ at points A and B , respectively. What is the length of AB ?



-  (a) The average of the digits of the integer 46 is 5. Including 46, how many two-digit positive integers have the average of their digits equal to 5?
  (b) When a decimal point is placed between the digits of the two-digit integer n , the resulting number is equal to the average of the digits of n . What is the value of n ?
  (c) The average of three positive integers is 28. When two additional integers, s and t , are included, the average of all five integers is 34. What is the average of s and t ?

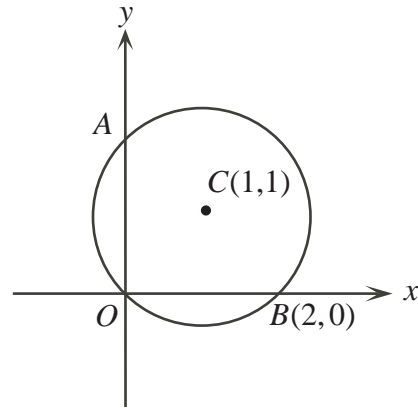
-  (a) Determine the coordinates of the vertex of the parabola $y = (x - 20)(x - 22)$.
  (b) Point A is the vertex of the parabola $y = x^2 + 2$, point B is the vertex of the parabola $y = x^2 - 6x + 7$, and O is the origin. Determine the area of $\triangle OAB$.

-  (a) In the diagram, the rectangle is divided into nine smaller rectangles. The areas of five of these rectangles are given. Determine the area of the rectangle labelled R .

3	1	
	2	R
5		10



- (b) In the diagram, the circle with centre $C(1,1)$ passes through the point $O(0,0)$, intersects the y -axis at A , and intersects the x -axis at $B(2,0)$. Determine, with justification, the coordinates of A and the area of the part of the circle that lies in the first quadrant.



5.



- (a) If a is chosen randomly from the set $\{1, 2, 3, 4, 5\}$ and b is chosen randomly from the set $\{6, 7, 8\}$, what is the probability that a^b is an even number?



- (b) A bag contains some blue and some green hats. On each turn, Julia removes one hat without looking, with each hat in the bag being equally likely to be chosen. If it is green, she adds a blue hat into the bag from her supply of extra hats, and if it is blue, she adds a green hat to the bag. The bag initially contains 4 blue hats and 2 green hats. What is the probability that the bag again contains 4 blue hats and 2 green hats after two turns?

6.



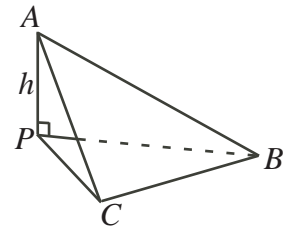
- (a) Suppose that, for some angles x and y ,

$$\begin{aligned}\sin^2 x + \cos^2 y &= \frac{3}{2}a \\ \cos^2 x + \sin^2 y &= \frac{1}{2}a^2\end{aligned}$$

Determine the possible value(s) of a .



- (b) Survivors on a desert island find a piece of plywood (ABC) in the shape of an equilateral triangle with sides of length 2 m. To shelter their goat from the sun, they place edge BC on the ground, lift corner A , and put in a vertical post PA which is h m long above ground. When the sun is directly overhead, the shaded region ($\triangle PBC$) on the ground directly underneath the plywood is an isosceles triangle with largest angle ($\angle BPC$) equal to 120° . Determine the value of h , to the nearest centimetre.



7.




- (a) The sequence 2, 5, 10, 50, 500, ... is formed so that each term after the second is the product of the two previous terms. The 15th term ends with exactly k zeroes. What is the value of k ?



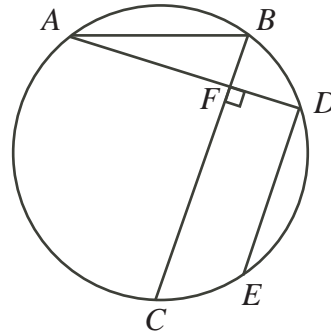
- (b) Suppose that a, b, c are three consecutive terms in an arithmetic sequence. Prove that $a^2 - bc$, $b^2 - ac$, and $c^2 - ab$ are also three consecutive terms in an arithmetic sequence.


(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7 is an arithmetic sequence with three terms.)


8.  (a) If $\log_2 x - 2 \log_2 y = 2$, determine y as a function of x , and sketch a graph of this function on the axes in the answer booklet.



- (b) In the diagram, AB and BC are chords of the circle with $AB < BC$. If D is the point on the circle such that AD is perpendicular to BC and E is the point on the circle such that DE is parallel to BC , carefully prove, explaining all steps, that $\angle EAC + \angle ABC = 90^\circ$.



9.  Define $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k .
- (a) Determine all real numbers k for which $f(x)$ is constant for all values of x .
- (b) If $k = -0.7$, determine all solutions to the equation $f(x) = 0$.
- (c) Determine all real numbers k for which there exists a real number c such that $f(c) = 0$.

10.  Points A_1, A_2, \dots, A_N are equally spaced around the circumference of a circle and $N \geq 3$. Three of these points are selected at random and a triangle is formed using these points as its vertices.
- (a) If $N = 7$, what is the probability that the triangle is acute? (A triangle is acute if each of its three interior angles is less than 90° .)
- (b) If $N = 2k$ for some positive integer $k \geq 2$, determine the probability that the triangle is acute.
- (c) If $N = 2k$ for some positive integer $k \geq 2$, determine all possible values of k for which the probability that the triangle is acute can be written in the form $\frac{a}{2007}$ for some positive integer a .



Canadian Mathematics Competition



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- Find your school results





Canadian Mathematics Competition

An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

Euclid Contest

for

The *CENTRE* for *EDUCATION* in *MATHEMATICS* and *COMPUTING*
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Tuesday, April 19, 2005

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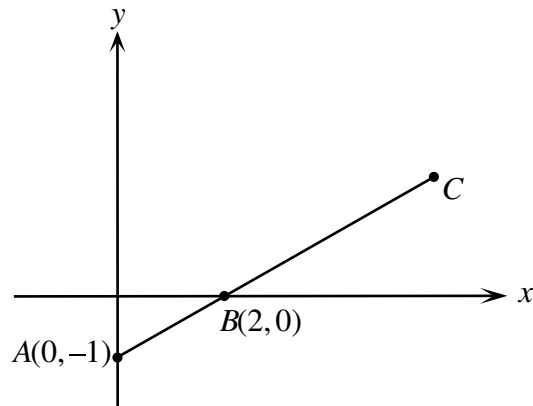
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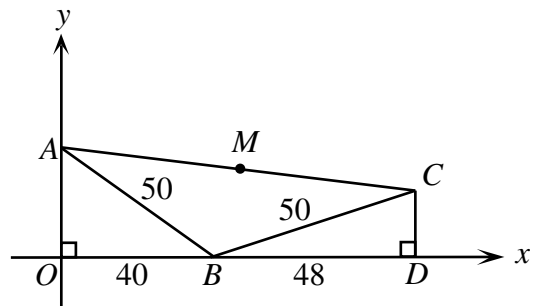
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1.  (a) If the point (a, a) lies on the line $3x - y = 10$, what is the value of a ?


 (b) In the diagram, points A , B and C lie on a line such that $BC = 2AB$. What are the coordinates of C ?



 (c) In the diagram, triangles AOB and CDB are right-angled and M is the midpoint of AC . What are the coordinates of M ?





2.  (a) If $y = 2x + 3$ and $4y = 5x + 6$, what is the value of x ?


 (b) If a , b and c are numbers such that


$$\begin{aligned} -3b + 7c &= -10 \\ b - 2c &= 3 \\ a + 2b - 5c &= 13 \end{aligned}$$

what is the value of a ?

 (c) John and Mary wrote the Euclid Contest. Two times John's score was 60 more than Mary's score. Two times Mary's score was 90 more than John's score. Determine the average of their two scores.

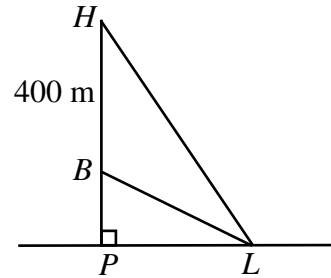
3.  (a) If $2^x = 2(16^{12}) + 2(8^{16})$, what is the value of x ?


 (b) If $f(x) = 2x - 1$, determine all real values of x such that $(f(x))^2 - 3f(x) + 2 = 0$.

4.  (a) Six tickets numbered 1 through 6 are placed in a box. Two tickets are randomly selected and removed together. What is the probability that the smaller of the two numbers on the tickets selected is less than or equal to 4?




- (b) A helicopter hovers at point H , directly above point P on level ground. Lloyd sits on the ground at a point L where $\angle HLP = 60^\circ$. A ball is dropped from the helicopter. When the ball is at point B , 400 m directly below the helicopter, $\angle BLP = 30^\circ$. What is the distance between L and P ?

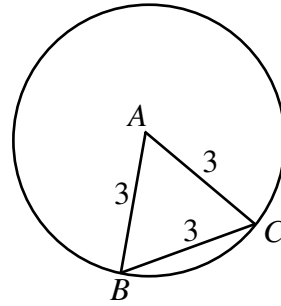



5.  (a) A goat starts at the origin $(0, 0)$ and then makes several moves. On move 1, it travels 1 unit up to $(0, 1)$. On move 2, it travels 2 units right to $(2, 1)$. On move 3, it travels 3 units down to $(2, -2)$. On move 4, it travels 4 units to $(-2, -2)$. It continues in this fashion, so that on move n , it turns 90° in a clockwise direction from its previous heading and travels n units in this new direction. After n moves, the goat has travelled a total of 55 units. Determine the coordinates of its position at this time.

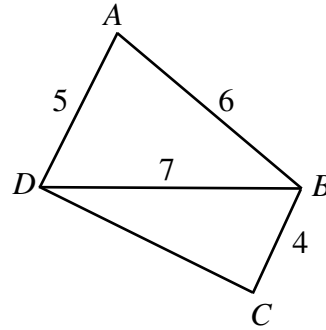



- (b) Determine all possible values of r such that the three term geometric sequence $4, 4r, 4r^2$ is also an arithmetic sequence.
(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9, 11 is an arithmetic sequence.)


6.  (a) Equilateral triangle ABC has side length 3, with vertices B and C on a circle of radius 3, as shown. The triangle is then rotated clockwise inside the circle: first about C until A reaches the circle, and then about A until B reaches the circle, and so on. Eventually the triangle returns to its original position and stops. What is the total distance travelled by the point B ?



-  (b) In the diagram, $ABCD$ is a quadrilateral in which $\angle A + \angle C = 180^\circ$. What is the length of CD ?

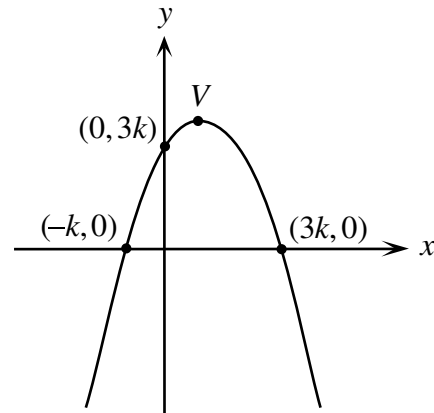



7.  (a) If $f(x) = \sin^2 x - 2 \sin x + 2$, what are the minimum and maximum values of $f(x)$?

-  (b) In the diagram, the parabola

$$y = -\frac{1}{4}(x - r)(x - s)$$


intersects the axes at three points. The vertex of this parabola is the point V . Determine the value of k and the coordinates of V .

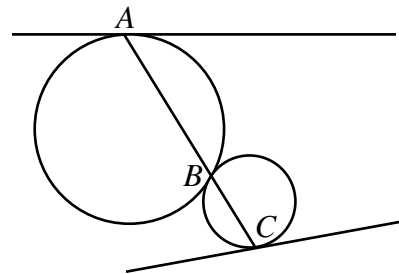



8.  (a) A function is defined by

$$f(x) = \begin{cases} 4 & \text{if } x < -4 \\ -x & \text{if } -4 \leq x \leq 5 \\ -5 & \text{if } x > 5 \end{cases}$$


On the grid in the answer booklet, sketch the graph $g(x) = \sqrt{25 - [f(x)]^2}$. State the shape of each portion of the graph.

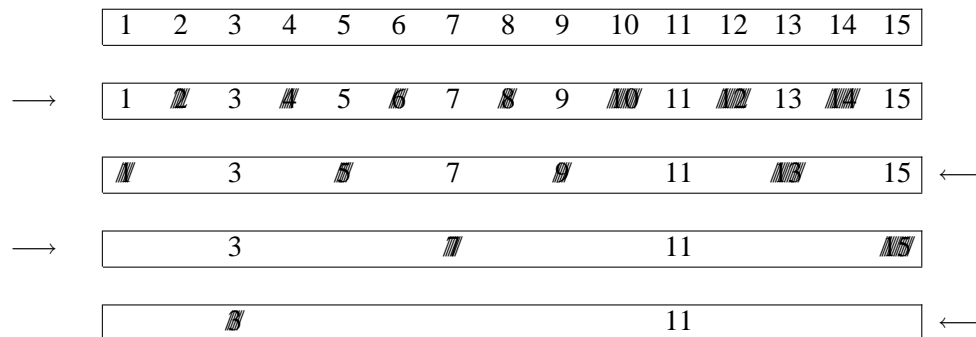
- (b)  In the diagram, two circles are tangent to each other at point B . A straight line is drawn through B cutting the two circles at A and C , as shown. Tangent lines are drawn to the circles at A and C . Prove that these two tangent lines are parallel.



9.  The circle $(x - p)^2 + y^2 = r^2$ has centre C and the circle $x^2 + (y - p)^2 = r^2$ has centre D . The circles intersect at two *distinct* points A and B , with x -coordinates a and b , respectively.

- (a) Prove that $a + b = p$ and $a^2 + b^2 = r^2$.
 (b) If r is fixed and p is then found to maximize the area of quadrilateral $CADB$, prove that either A or B is the origin.
 (c) If p and r are integers, determine the minimum possible distance between A and B . Find positive integers p and r , each larger than 1, that give this distance.

10.  A school has a row of n open lockers, numbered 1 through n . After arriving at school one day, Josephine starts at the beginning of the row and closes every second locker until reaching the end of the row, as shown in the example below. Then on her way back, she closes every second locker that is still open. She continues in this manner along the row, until only one locker remains open. Define $f(n)$ to be the number of the last open locker. For example, if there are 15 lockers, then $f(15) = 11$ as shown below:



- (a) Determine $f(50)$.
 (b) Prove that there is no positive integer n such that $f(n) = 2005$.
 (c) Prove that there are infinitely many positive integers n such that $f(n) = f(2005)$.



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Euclid Contest

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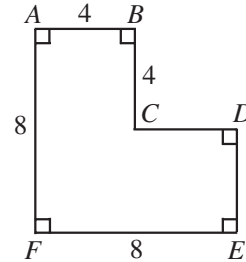
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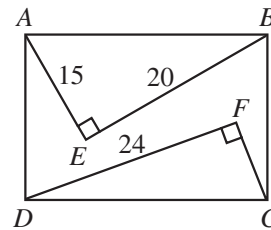
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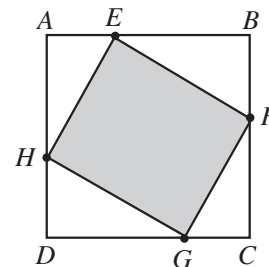
1.  (a) In the diagram, what is the area of figure $ABCDEF$?





-  (b) In the diagram, $ABCD$ is a rectangle with $AE = 15$, $EB = 20$ and $DF = 24$. What is the length of CF ?

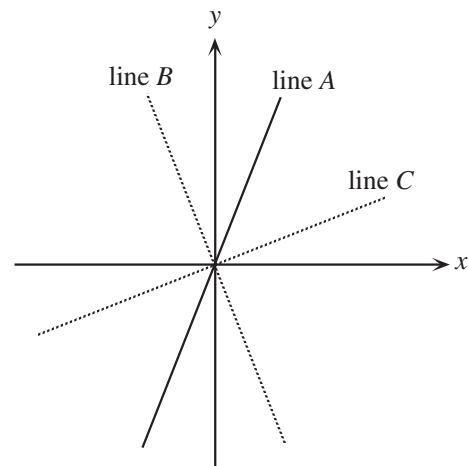


-  (c) In the diagram, $ABCD$ is a square of side length 6. Points E , F , G , and H are on AB , BC , CD , and DA , respectively, so that the ratios $AE:EB$, $BF:FC$, $CG:GD$, and $DH:HA$ are all equal to 1:2. What is the area of $EFGH$?



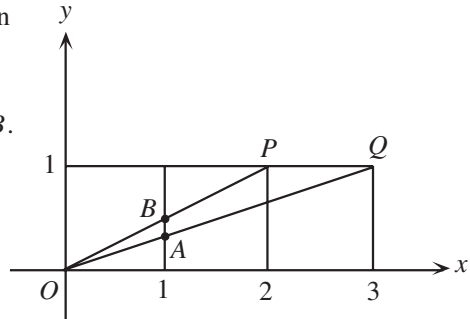
2.  (a) A horizontal line has the same y -intercept as the line $3x - y = 6$. What is the equation of this horizontal line?


-  (b) In the diagram, line A has equation $y = 2x$. Line B is obtained by reflecting line A in the y -axis. Line C is perpendicular to line B . What is the slope of line C ?






- (c) Three squares, each of side length 1, are drawn side by side in the first quadrant, as shown. Lines are drawn from the origin to P and Q . Determine, with explanation, the length of AB .



3.  (a) In an arithmetic sequence with five terms, the sum of the first two terms is 2 and the sum of the last two terms is -18 . What is the third term of this sequence? (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9, 11 is an arithmetic sequence with five terms.)




- (b) If $x - y = 4\sqrt{2}$ and $xy = 56$, determine the two possible values of $x + y$.

4.  (a) Two fair dice, each having six faces numbered 1 to 6, are thrown. What is the probability that the *product* of the two numbers on the top faces is divisible by 5?

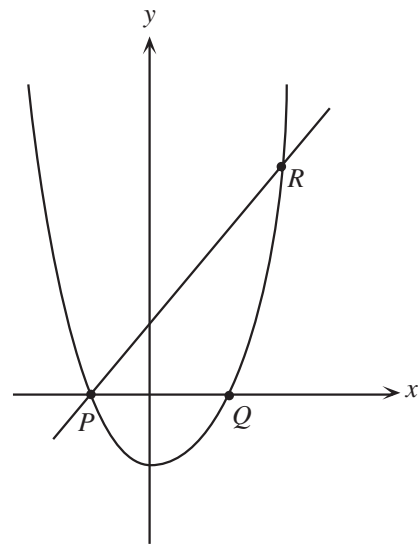



- (b) If $f(x) = x^2 - x + 2$, $g(x) = ax + b$, and $f(g(x)) = 9x^2 - 3x + 2$, determine all possible ordered pairs (a, b) which satisfy this relationship.

5.  (a) If $16^x = 2^{x+5} - 2^{x+4}$, determine the value of x .



- (b) In the diagram, the parabola with equation $y = x^2 + tx - 2$ intersects the x -axis at points P and Q . Also, the line with equation $y = 3x + 3$ intersects the parabola at points P and R . Determine the value of t and the area of triangle PQR .



6.  (a) Lori has a loonie, three quarters, three dimes, three nickels, and five pennies. She wishes to purchase a toy helicopter for exactly \$1.34. What is the maximum number of coins that she can use to make the purchase? (In Canada, a loonie is worth \$1.00, a quarter is worth \$0.25, a dime is worth \$0.10, a nickel is worth \$0.05, and a penny is worth \$0.01.)




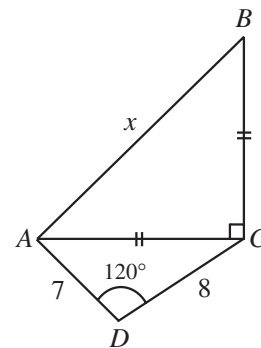
- (b) Digital images consist of a very large number of equally spaced dots called *pixels*. The *resolution* of an image is the number of pixels/cm in each of the horizontal and vertical directions.

Thus, an image with dimensions 10 cm by 15 cm and a resolution of 75 pixels/cm has a total of $(10 \times 75) \times (15 \times 75) = 843\,750$ pixels.

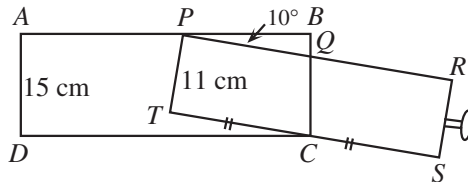
If each of these dimensions was increased by $n\%$ and the resolution was decreased by $n\%$, the image would have 345 600 pixels.


Determine the value of n .

7.  (a) In the diagram, $AC = BC$, $AD = 7$, $DC = 8$, and $\angle ADC = 120^\circ$. What is the value of x ?



- (b) In the diagram, a drawer, $PRST$, that is 11 cm high is in a long slot, $ABCD$, which is 15 cm high. The drawer is pulled out so that the midpoint of its base rests at C . The drawer is tilted so that the top back edge of the drawer touches the top of the slot. If the angle between the drawer and the slot is 10° , determine the length of the drawer, to the nearest tenth of a centimetre.



8.  (a) If $T = x^2 + \frac{1}{x^2}$, determine the values of b and c so that $x^6 + \frac{1}{x^6} = T^3 + bT + c$ for all non-zero real numbers x .

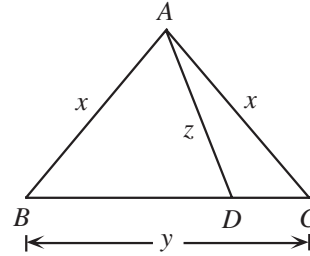



- (b) If x is a real number satisfying $x^3 + \frac{1}{x^3} = 2\sqrt{5}$, determine the exact value of $x^2 + \frac{1}{x^2}$.

9.  A *Kirk triplet* is a triple (x, y, z) of integers such that:

- i) $x > z$,
- ii) z is a prime number, and
- iii) there is a triangle ABC with $AB = AC = x$, $BC = y$, and a point D on BC such that $AD = z$ and $\angle ADB = 60^\circ$.

- (a) Find the Kirk triplet with $x = 7$ and $z = 5$.
- (b) Determine all other Kirk triplets with $z = 5$.
- (c) Determine the Kirk triplet for which $\cos(\angle ABC)$ is as close to 0.99 as possible.



10.  A *Skolem sequence* of order n is a sequence $(s_1, s_2, \dots, s_{2n})$ of $2n$ integers satisfying the conditions:

- i) for every k in $\{1, 2, 3, \dots, n\}$, there exist exactly two elements s_i and s_j with $s_i = s_j = k$, and
- ii) if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

For example, $(4, 2, 3, 2, 4, 3, 1, 1)$ is a Skolem sequence of order 4.

- (a) List all Skolem sequences of order 4.
- (b) Determine, with justification, all Skolem sequences of order 9 which satisfy all of the following three conditions:
 - I) $s_3 = 1$,
 - II) $s_{18} = 8$, and
 - III) between any two equal even integers, there is exactly one odd integer.
- (c) Prove that there is no Skolem sequence of order n , if n is of the form $4k + 2$ or $4k + 3$, where k is a non-negative integer.

PUBLICATIONS

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Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING Awards

Tuesday, April 15, 2003

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Time: $2\frac{1}{2}$ hours

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Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

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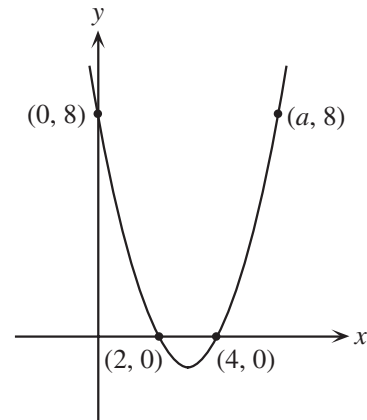
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3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

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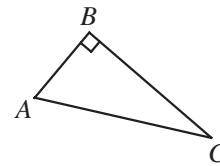
1. 💡 (a) In the diagram, the parabola cuts the y -axis at the point $(0, 8)$, cuts the x -axis at the points $(2, 0)$ and $(4, 0)$, and passes through the point $(a, 8)$. What is the value of a ?



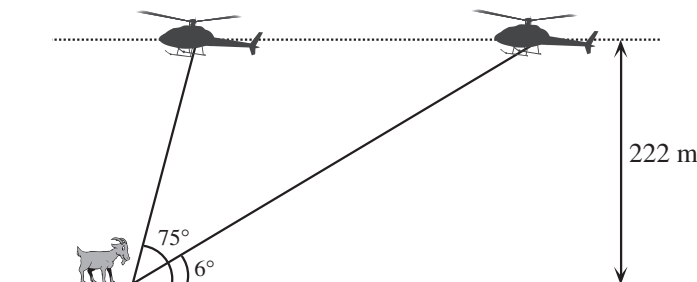
- 💡 (b) The quadratic equation $x^2 + 6x + k = 0$ has two equal roots. What is the value of k ?
- 🔍 (c) The line $y = 2x + 2$ intersects the parabola $y = x^2 - 3x + c$ at two points. One of these points is $(1, 4)$. Determine the coordinates of the second point of intersection.


2. 💡 (a) If $0^\circ < x < 90^\circ$ and $3\sin(x) - \cos(15^\circ) = 0$, what is the value of x to the nearest tenth of a degree?

- 💡 (b) In the diagram, $\triangle ABC$ is right-angled at B and $AC = 20$. If $\sin C = \frac{3}{5}$, what is the length of side BC ?



- 🔍 (c) A helicopter is flying due west over level ground at a constant altitude of 222 m and at a constant speed. A lazy, stationary goat, which is due west of the helicopter, takes two measurements of the angle between the ground and the helicopter. The first measurement the goat makes is 6° and the second measurement, which he makes 1 minute later, is 75° . If the helicopter has not yet passed over the goat, as shown, how fast is the helicopter travelling to the nearest kilometre per hour?



3.  (a) The function $f(x)$ has the property that $f(2x+3) = 2f(x) + 3$ for all x . If $f(0) = 6$, what is the value of $f(9)$?




- (b) Suppose that the functions $f(x)$ and $g(x)$ satisfy the system of equations

$$f(x) + 3g(x) = x^2 + x + 6$$


$$2f(x) + 4g(x) = 2x^2 + 4$$

for all x . Determine the values of x for which $f(x) = g(x)$.

4.  (a) In a short-track speed skating event, there are five finalists including two Canadians. The first three skaters to finish the race win a medal. If all finalists have the same chance of finishing in any position, what is the probability that neither Canadian wins a medal?




- (b) Determine the number of positive integers less than or equal to 300 that are multiples of 3 or 5, but are not multiples of 10 or 15.

5.  (a) In the series of odd numbers $1 + 3 + 5 - 7 - 9 - 11 + 13 + 15 + 17 - 19 - 21 - 23 \dots$ the signs alternate every three terms, as shown. What is the sum of the first 300 terms of the series?



- (b) A two-digit number has the property that the square of its tens digit plus ten times its units digit equals the square of its units digit plus ten times its tens digit. Determine all two-digit numbers which have this property, and are prime numbers.


6.  (a) A lead box contains samples of two radioactive isotopes of iron. Isotope A decays so that after every 6 minutes, the number of atoms remaining is halved. Initially, there are twice as many atoms of isotope A as of isotope B, and after 24 minutes there are the same number of atoms of each isotope. How long does it take the number of atoms of isotope B to halve?

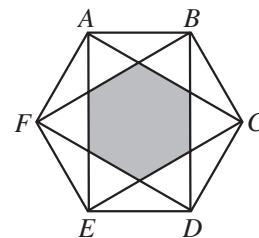


- (b) Solve the system of equations:

$$\log_{10}(x^3) + \log_{10}(y^2) = 11$$

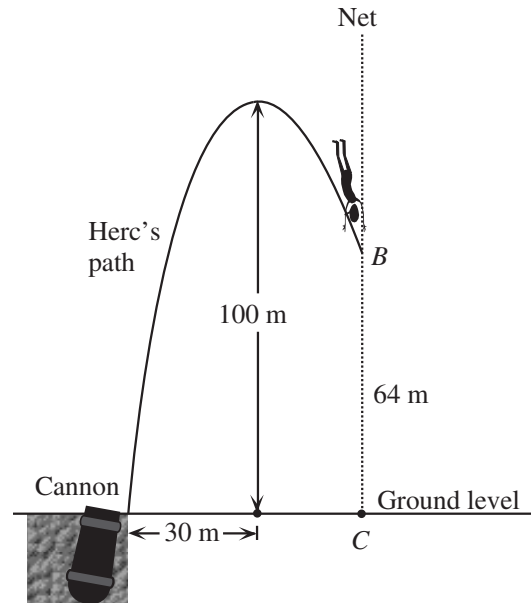
$$\log_{10}(x^2) - \log_{10}(y^3) = 3$$

7.  (a) A regular hexagon is a six-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, $ABCDEF$ is a regular hexagon with an area of 36. The region common to the equilateral triangles ACE and BDF is a hexagon, which is shaded as shown. What is the area of the shaded hexagon?

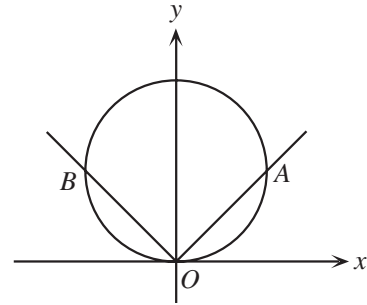




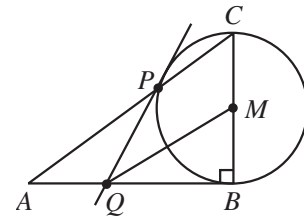
- (b) At the Big Top Circus, Herc the Human Cannonball is fired out of the cannon at ground level. (For the safety of the spectators, the cannon is partially buried in the sand floor.) Herc's trajectory is a parabola until he catches the vertical safety net, on his way down, at point B . Point B is 64 m directly above point C on the floor of the tent. If Herc reaches a maximum height of 100 m, directly above a point 30 m from the cannon, determine the horizontal distance from the cannon to the net.




8. (a) A circle with its centre on the y -axis intersects the graph of $y = |x|$ at the origin, O , and exactly two other distinct points, A and B , as shown. Prove that the ratio of the area of triangle ABO to the area of the circle is always $1 : \pi$.



- (b) In the diagram, triangle ABC has a right angle at B and M is the midpoint of BC . A circle is drawn using BC as its diameter. P is the point of intersection of the circle with AC . The tangent to the circle at P cuts AB at Q . Prove that QM is parallel to AC .



9. Cyclic quadrilateral $ABCD$ has $AB = AD = 1$, $CD = \cos \angle ABC$, and $\cos \angle BAD = -\frac{1}{3}$. Prove that BC is a diameter of the circumscribed circle.

10.  A positive integer n is called “savage” if the integers $\{1, 2, \dots, n\}$ can be partitioned into three sets A , B and C such that
- i) the sum of the elements in each of A , B , and C is the same,
 - ii) A contains only odd numbers,
 - iii) B contains only even numbers, and
 - iv) C contains every multiple of 3 (and possibly other numbers).
- (a) Show that 8 is a savage integer.
- (b) Prove that if n is an even savage integer, then $\frac{n+4}{12}$ is an integer.
- (c) Determine all even savage integers less than 100.

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Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING Awards

Tuesday, April 16, 2002

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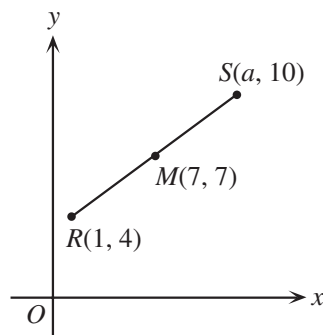
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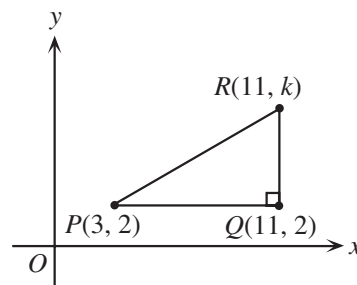
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1. 💡 (a) If $M(7,7)$ is the midpoint of the line segment which joins $R(1,4)$ and $S(a,10)$, what is the value of a ?

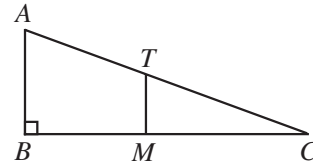



- 💡 (b) In the diagram, points $P(3,2)$, $Q(11,2)$ and $R(11,k)$ form a triangle with area 24, where $k > 0$. What is the value of k ?

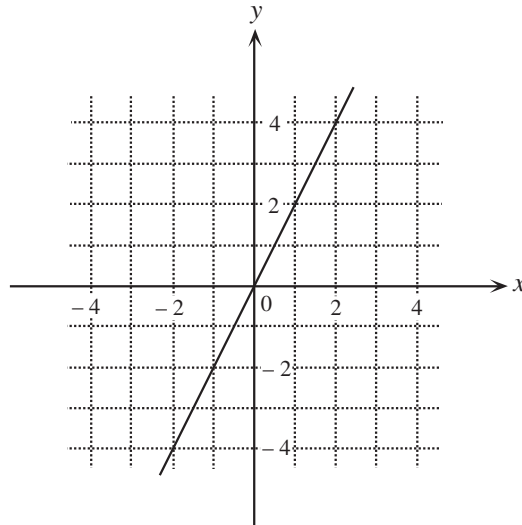


- 📐 (c) Lines are *concurrent* if they each pass through the same point. The lines $y = 2x + 3$, $y = 8x + 15$, and $y = 5x + b$ are concurrent. What is the value of b ?
2. 💡 (a) The quadratic equation $x^2 - 3x + c = 0$ has $x = 4$ as one of its roots. What is its second root?
- 💡 (b) The rational expression $\frac{2x^2 + 1}{x^2 - 3}$ may be written as $2 + \frac{A}{x^2 - 3}$, where A is an integer. What is the value of A ?
- 📐 (c) The parabola $y = x^2 - 4x + 3$ is translated 5 units to the right. In this new position, the equation of the parabola is $y = x^2 - 14x + d$. Determine the value of d .
3. 💡 (a) Three bins are labelled A, B and C, and each bin contains four balls numbered 1, 2, 3, and 4. The balls in each bin are mixed, and then a student chooses one ball at random from each of the bins. If a , b and c are the numbers on the balls chosen from bins A, B and C, respectively, the student wins a toy helicopter when $a = b + c$. There are 64 ways to choose the three balls. What is the probability that the student wins the prize?
- 📐 (b) Three positive integers a , ar and ar^2 form an increasing sequence. If the product of the three integers in this sequence is 216, determine all sequences satisfying the given conditions.


4.  (a) In the diagram, triangle ABC is right-angled at B . MT is the perpendicular bisector of BC with M on BC and T on AC . If $AT = AB$, what is the size of $\angle ACB$?




-  (b) The graph of $y = f(x)$, where $f(x) = 2x$, is given on the grid below.



- (i) On the grid in the answer booklet, draw and label the graphs of the inverse function $y = f^{-1}(x)$ and the reciprocal function $y = \frac{1}{f(x)}$.
- (ii) State the coordinates of the points where $f^{-1}(x) = \frac{1}{f(x)}$.
- (iii) Determine the numerical value of $f^{-1}\left(\frac{1}{f\left(\frac{1}{2}\right)}\right)$.

5.  (a) What are all values of x such that $\log_5(x+3) + \log_5(x-1) = 1$?


-  (b) A chef aboard a luxury liner wants to cook a goose. The time t in hours to cook a goose at 180°C depends on the mass of the goose m in kilograms according to the formula

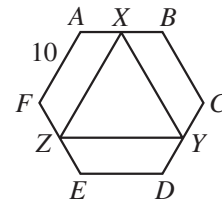
$$t = am^b$$


where a and b are constants. The table below gives the times observed to cook a goose at 180°C .


Mass, m (kg)	Time, t (h)
3.00	2.75
6.00	3.75

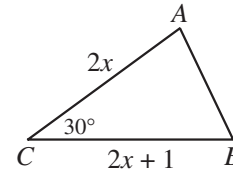
- (i) Using the data in the table, determine both a and b to two decimal places.
- (ii) Suppose that the chef wants to cook a goose with a mass of 8.00 kg at 180°C . How long will it take until his goose is cooked?


6.  (a) In the diagram, $ABCDEF$ is a regular hexagon with a side length of 10. If X , Y and Z are the midpoints of AB , CD and EF , respectively, what is the length of XZ ?

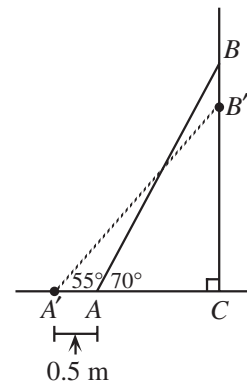



-  (b) A circle passes through the origin and the points of intersection of the parabolas $y = x^2 - 3$ and $y = -x^2 - 2x + 9$. Determine the coordinates of the centre of this circle.

7.  (a) In the diagram, $AC = 2x$, $BC = 2x + 1$ and $\angle ACB = 30^\circ$. If the area of $\triangle ABC$ is 18, what is the value of x ?





-  (b) A ladder, AB , is positioned so that its bottom sits on horizontal ground and its top rests against a vertical wall, as shown. In this initial position, the ladder makes an angle of 70° with the horizontal. The bottom of the ladder is then pushed 0.5 m away from the wall, moving the ladder to position $A'B'$. In this new position, the ladder makes an angle of 55° with the horizontal. Calculate, to the nearest centimetre, the distance that the ladder slides down the wall (that is, the length of BB').

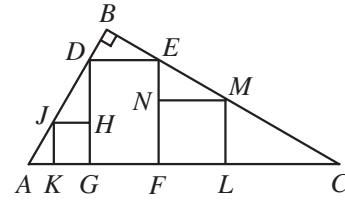




8.  (a) In a soccer league with 5 teams, each team plays 20 games (that is, 5 games with each of the other 4 teams). For each team, every game ends in a win (W), a loss (L), or a tie (T). The numbers of wins, losses and ties for each team at the end of the season are shown in the table. Determine the values of x , y and z .

Team	W	L	T
A	2	15	3
B	7	9	4
C	6	12	2
D	10	8	2
E	x	y	z

-  (b) Prove that it is not possible to create a sequence of 4 numbers a, b, c, d , such that the sum of any two consecutive terms is positive, and the sum of any three consecutive terms is negative.

9.  (a) In triangle ABC , $\angle ABC = 90^\circ$. Rectangle $DEFG$ is inscribed in $\triangle ABC$, as shown. Squares $JKGH$ and $MLFN$ are inscribed in $\triangle AGD$ and $\triangle CFE$, respectively. If the side length of $JHGK$ is v , the side length of $MLFN$ is w , and $DG = u$, prove that $u = v + w$.



-  (b) Three thin metal rods of lengths 9, 12 and 15 are welded together to form a right-angled triangle, which is held in a horizontal position. A solid sphere of radius 5 rests in the triangle so that it is tangent to each of the three sides. Assuming that the thickness of the rods can be neglected, how high above the plane of the triangle is the top of the sphere?
10.  A triangle is called *Heronian* if each of its side lengths is an integer and its area is also an integer. A triangle is called *Pythagorean* if it is right-angled and each of its side lengths is an integer.
- Show that every Pythagorean triangle is Heronian.
 - Show that every odd integer greater than 1 is a side length of some Pythagorean triangle.
 - Find a Heronian triangle which has all side lengths different, and no side length divisible by 3, 5, 7 or 11.

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Copies of previous contests, together with solutions, are available in a variety of packages, as described below. Please order by package number, and note that the number defines the competition and the number of papers included. Each package has two numbers. Numbers prefixed with E are English language supplies – Numbers prefixed with F are French language supplies. Each package is considered as one title.

(For copies of 1999 - 2001 contests and solutions, please see our website (<http://www.cemc.uwaterloo.ca>)).

Included is one copy of any one contest, together with solutions, for each of 1998, 1999, and 2000. Recommended for individuals.

E 213, F 213	Gauss Contest (Grades 7, 8)	\$10.00
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An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

Thursday, April 19, 2001

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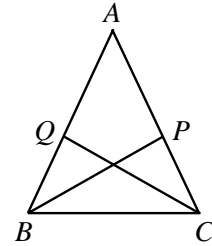
1. 💡 (a) What are the values of x such that $(2x - 3)^2 = 9$?

💡 (b) If $f(x) = x^2 - 3x - 5$, what are the values of k such that $f(k) = k$?

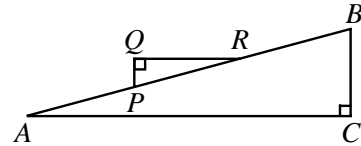
🔍 (c) Determine all (x, y) such that $x^2 + y^2 = 25$ and $x - y = 1$.

2. 💡 (a) The vertex of the parabola $y = (x - b)^2 + b + h$ has coordinates $(2, 5)$. What is the value of h ?

💡 (b) In the isosceles triangle ABC , $AB = AC$ and $\angle BAC = 40^\circ$. Point P is on AC such that BP is the bisector of $\angle ABC$. Similarly, Q is on AB such that CQ bisects $\angle ACB$. What is the size of $\angle APB$, in degrees?





🔍 (c) In the diagram, $AB = 300$, $PQ = 20$, and $QR = 100$. Also, QR is parallel to AC . Determine the length of BC , to the nearest integer.

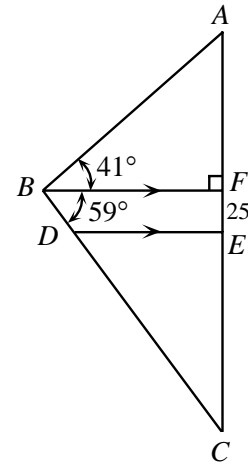



3. 💡 (a) In an increasing sequence of numbers with an odd number of terms, the difference between any two consecutive terms is a constant d , and the middle term is 302. When the last 4 terms are removed from the sequence, the middle term of the resulting sequence is 296. What is the value of d ?


🔍 (b) There are two increasing sequences of five consecutive integers, each of which have the property that the sum of the squares of the first three integers in the sequence equals the sum of the squares of the last two. Determine these two sequences.


4.  (a) If $f(t) = \sin\left(\pi t - \frac{\pi}{2}\right)$, what is the smallest positive value of t at which $f(t)$ attains its minimum value?

-  (b) In the diagram, $\angle ABF = 41^\circ$, $\angle CBF = 59^\circ$, DE is parallel to BF , and $EF = 25$. If $AE = EC$, determine the length of AE , to 2 decimal places.



5.  (a) Determine all integer values of x such that $(x^2 - 3)(x^2 + 5) < 0$.


-  (b) At present, the sum of the ages of a husband and wife, P , is six times the sum of the ages of their children, C . Two years ago, the sum of the ages of the husband and wife was ten times the sum of the ages of the same children. Six years from now, it will be three times the sum of the ages of the same children. Determine the number of children.

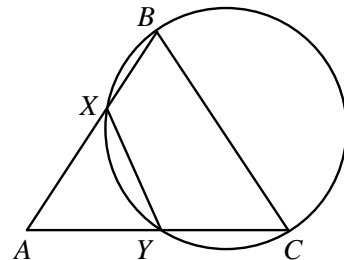
6.  (a) Four teams, A , B , C , and D , competed in a field hockey tournament. Three coaches predicted who would win the Gold, Silver and Bronze medals:


Medal	Gold	Silver	Bronze
Team			


- Coach 1 predicted Gold for A , Silver for B , and Bronze for C ,
- Coach 2 predicted Gold for B , Silver for C , and Bronze for D ,
- Coach 3 predicted Gold for C , Silver for A , and Bronze for D .




Each coach predicted exactly one medal winner correctly. Complete the table **in the answer booklet** to show which team won which medal.


-  (b) In triangle ABC , $AB = BC = 25$ and $AC = 30$. The circle with diameter BC intersects AB at X and AC at Y . Determine the length of XY .

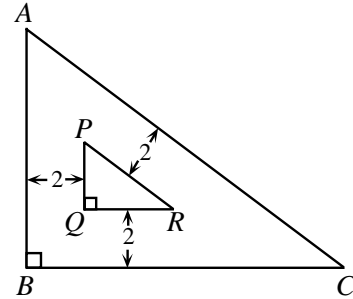



7.  (a) What is the value of x such that $\log_2(\log_2(2x - 2)) = 2$?

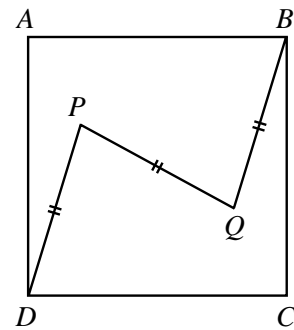
-  (b) Let $f(x) = 2^{kx} + 9$, where k is a real number. If $f(3) : f(6) = 1 : 3$, determine the value of $f(9) - f(3)$.

8.  (a) On the grid provided in the answer booklet, sketch $y = x^2 - 4$ and $y = 2|x|$.
-  (b) Determine, with justification, all values of k for which $y = x^2 - 4$ and $y = 2|x| + k$ do **not** intersect.
-  (c) State the values of k for which $y = x^2 - 4$ and $y = 2|x| + k$ intersect in exactly two points. (Justification is not required.)

9.  Triangle ABC is right-angled at B and has side lengths which are integers. A second triangle, PQR , is located inside $\triangle ABC$ as shown, such that its sides are parallel to the sides of $\triangle ABC$ and the distance between parallel lines is 2. Determine the side lengths of all possible triangles ABC , such that the area of $\triangle ABC$ is 9 times that of $\triangle PQR$.



10.  Points P and Q are located inside the square $ABCD$ such that DP is parallel to QB and $DP = QB = PQ$. Determine the minimum possible value of $\angle ADP$.





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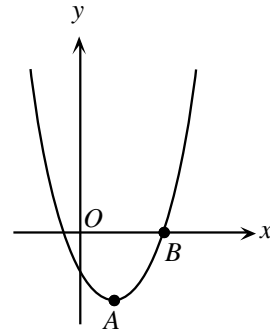
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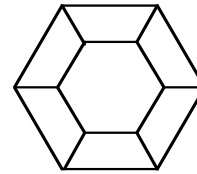
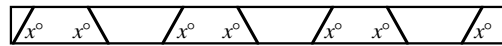
1.  (a) If $x + 27^{\frac{1}{3}} = 125^{\frac{1}{3}}$, what is the value of x ?


 (b) The line $y = ax + c$ is parallel to the line $y = 2x$ and passes through the point $(1, 5)$. What is the value of c ?


 (c) The parabola with equation $y = (x - 2)^2 - 16$ has its vertex at A and intersects the x -axis at B , as shown. Determine the equation for the line passing through A and B .




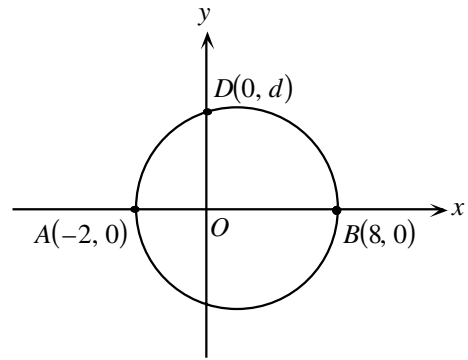
2.  (a) Six identical pieces are cut from a board, as shown in the diagram. The angle of each cut is x° . The pieces are assembled to form a hexagonal picture frame as shown. What is the value of x ?




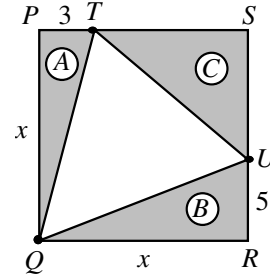
 (b) If $\log_{10} x = 3 + \log_{10} y$, what is the value of $\frac{x}{y}$?


 (c) If $x + \frac{1}{x} = \frac{13}{6}$, determine all values of $x^2 + \frac{1}{x^2}$.


3.  (a) A circle, with diameter AB as shown, intersects the positive y -axis at point $D(0, d)$. Determine d .



-  (b) A square $PQRS$ with side of length x is subdivided into four triangular regions as shown so that area $(A) + \text{area } (B) = \text{area } (C)$. If $PT = 3$ and $RU = 5$, determine the value of x .




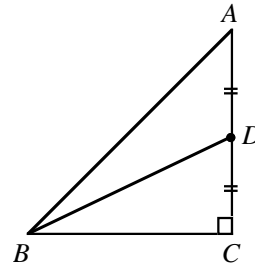
4.  (a) A die, with the numbers 1, 2, 3, 4, 6, and 8 on its six faces, is rolled. After this roll, if an odd number appears on the top face, all odd numbers on the die are doubled. If an even number appears on the top face, all the even numbers are halved. If the given die changes in this way, what is the probability that a 2 will appear on the second roll of the die?


-  (b) The table below gives the final standings for seven of the teams in the English Cricket League in 1998. At the end of the year, each team had played 17 matches and had obtained the total number of points shown in the last column. Each win W , each draw D , each bonus bowling point A , and each bonus batting point B received w , d , a and b points respectively, where w , d , a and b are positive integers. *No points* are given for a loss. Determine the values of w , d , a and b if total points awarded are given by the formula: $\text{Points} = w \times W + d \times D + a \times A + b \times B$.

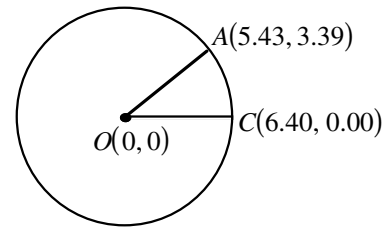
Final Standings


	W	$Losses$	D	A	B	$Points$
Sussex	6	7	4	30	63	201
Warks	6	8	3	35	60	200
Som	6	7	4	30	54	192
Derbys	6	7	4	28	55	191
Kent	5	5	7	18	59	178
Worcs	4	6	7	32	59	176
Glam	4	6	7	36	55	176


5.  (a) In the diagram, $AD = DC$, $\sin \angle DBC = 0.6$ and $\angle ACB = 90^\circ$. What is the value of $\tan \angle ABC$?

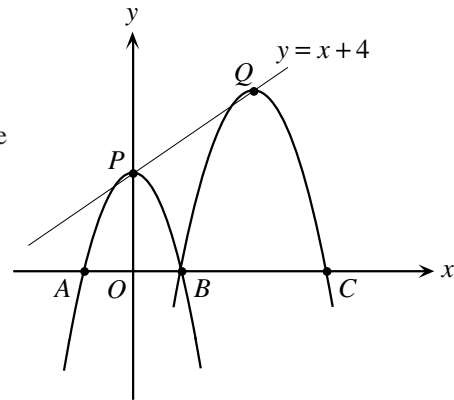



-  (b) On a cross-sectional diagram of the Earth, the x and y -axes are placed so that $O(0, 0)$ is the centre of the Earth and $C(6.40, 0.00)$ is the location of Cape Canaveral. A space shuttle is forced to land on an island at $A(5.43, 3.39)$, as shown. Each unit represents 1 000 km. Determine the distance from Cape Canaveral to the island, measured on the surface of the earth, to the nearest 10 km.




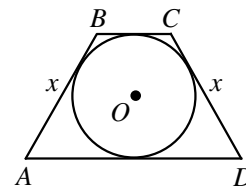
6.  (a) Let $\lfloor x \rfloor$ represent the greatest integer which is less than or equal to x . For example, $\lfloor 3 \rfloor = 3$, $\lfloor 2.6 \rfloor = 2$. If x is positive and $x \lfloor x \rfloor = 17$, what is the value of x ?


-  (b) The parabola $y = -x^2 + 4$ has vertex P and intersects the x -axis at A and B . The parabola is translated from its original position so that its vertex moves along the line $y = x + 4$ to the point Q . In this position, the parabola intersects the x -axis at B and C . Determine the coordinates of C .

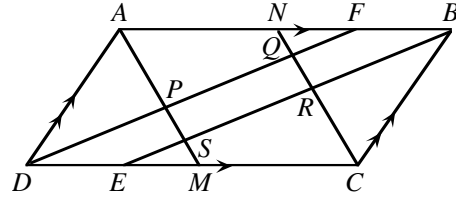



7.  (a) A cube has edges of length n , where n is an integer. Three faces, meeting at a corner, are painted red. The cube is then cut into n^3 smaller cubes of unit length. If exactly 125 of these cubes have no faces painted red, determine the value of n .


-  (b) In the isosceles trapezoid $ABCD$, $AB = CD = x$. The area of the trapezoid is 80 and the circle with centre O and radius 4 is tangent to the four sides of the trapezoid. Determine the value of x .

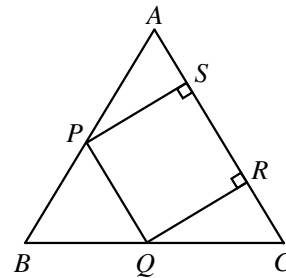


8.  In parallelogram $ABCD$, $AB = a$ and $BC = b$, where $a > b$. The points of intersection of the angle bisectors are the vertices of quadrilateral $PQRS$.
- (a) Prove that $PQRS$ is a rectangle.
- (b) Prove that $PR = a - b$.



9.  A permutation of the integers $1, 2, \dots, n$ is a listing of these integers in some order. For example, $(3, 1, 2)$ and $(2, 1, 3)$ are two different permutations of the integers $1, 2, 3$. A permutation (a_1, a_2, \dots, a_n) of the integers $1, 2, \dots, n$ is said to be “fantastic” if $a_1 + a_2 + \dots + a_k$ is divisible by k , for **each** k from 1 to n . For example, $(3, 1, 2)$ is a fantastic permutation of $1, 2, 3$ because 3 is divisible by 1 , $3 + 1$ is divisible by 2 , and $3 + 1 + 2$ is divisible by 3 . However, $(2, 1, 3)$ is not fantastic because $2 + 1$ is not divisible by 2 .
- (a) Show that no fantastic permutation exists for $n = 2000$.
- (b) Does a fantastic permutation exist for $n = 2001$? Explain.

10.  An equilateral triangle ABC has side length 2 . A square, $PQRS$, is such that P lies on AB , Q lies on BC , and R and S lie on AC as shown. The points P, Q, R , and S move so that P, Q and R always remain on the sides of the triangle and S moves from AC to AB through the interior of the triangle. If the points P, Q, R and S always form the vertices of a square, show that the path traced out by S is a straight line parallel to BC .





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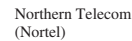
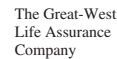
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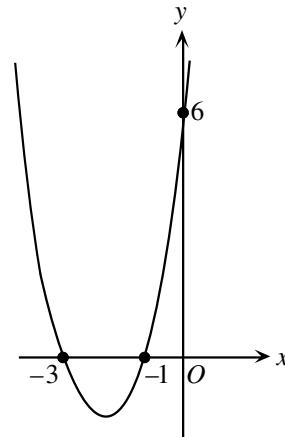
 (b) If the point $P(-3, 2)$ is on the line $3x + 7ky = 5$, what is the value of k ?


 (c) If $x^2 - x - 2 = 0$, determine all possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.

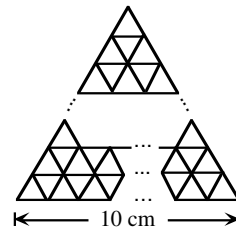
2.  (a) The circle defined by the equation $(x + 4)^2 + (y - 3)^2 = 9$ is moved horizontally until its centre is on the line $x = 6$. How far does the centre of the circle move?


 (b) The parabola defined by the equation $y = (x - 1)^2 - 4$ intersects the x -axis at the points P and Q . If (a, b) is the mid-point of the line segment PQ , what is the value of a ?


 (c) Determine an equation of the quadratic function shown in the diagram.

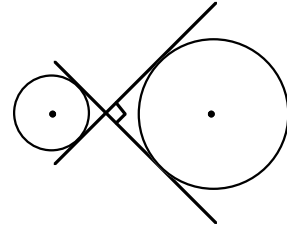



3.  (a) How many equilateral triangles of side 1 cm, placed as shown in the diagram, are needed to completely cover the interior of an equilateral triangle of side 10 cm?




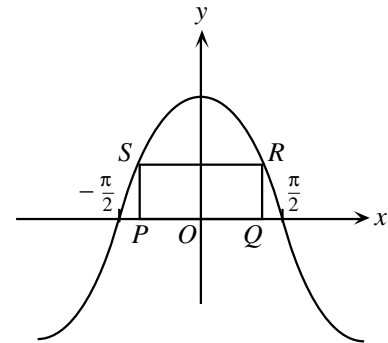
 (b) The populations of Alphaville and Betaville were equal at the end of 1995. The population of Alphaville decreased by 2.9% during 1996, then increased by 8.9% during 1997, and then increased by 6.9% during 1998. The population of Betaville increased by $r\%$ in each of the three years. If the populations of the towns are equal at the end of 1998, determine the value of r correct to one decimal place.


4.  (a) In the diagram, the tangents to the two circles intersect at 90° as shown. If the radius of the smaller circle is 2, and the radius of the larger circle is 5, what is the distance between the centres of the two circles?

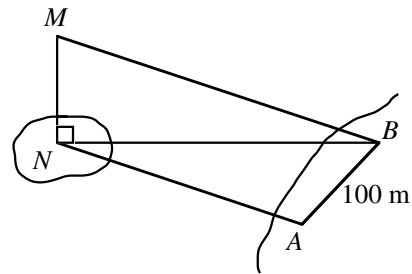



-  (b) A circular ferris wheel has a radius of 8 m and rotates at a rate of 12° per second. At $t = 0$, a seat is at its lowest point which is 2 m above the ground. Determine how high the seat is above the ground at $t = 40$ seconds.

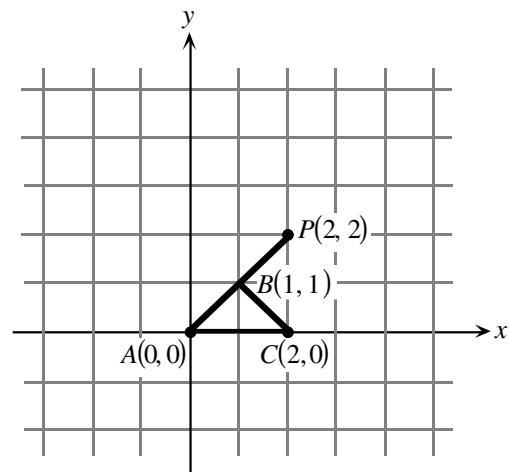
5.  (a) A rectangle $PQRS$ has side PQ on the x -axis and touches the graph of $y = k \cos x$ at the points S and R as shown. If the length of PQ is $\frac{\pi}{3}$ and the area of the rectangle is $\frac{5\pi}{3}$, what is the value of k ?




-  (b) In determining the height, MN , of a tower on an island, two points A and B , 100 m apart, are chosen on the same horizontal plane as N . If $\angle NAB = 108^\circ$, $\angle ABN = 47^\circ$ and $\angle MBN = 32^\circ$, determine the height of the tower to the nearest metre.



6.  (a) The points A , P and a third point Q (not shown) are the vertices of a triangle which is similar to triangle ABC . What are the coordinates of all possible positions for Q ?

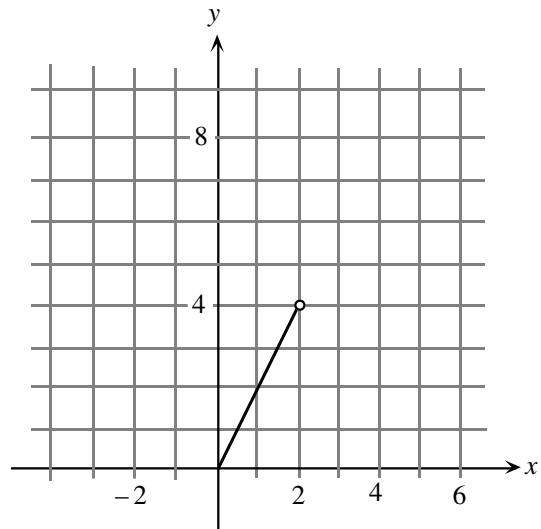


-  (b) Determine the coordinates of the points of intersection of the graphs of $y = \log_{10}(x - 2)$ and $y = 1 - \log_{10}(x + 1)$.

7.  (a) On the grid provided in the answer booklet, draw the graphs of the functions $y = -2\sqrt{x+1}$ and $y = \sqrt{x-2}$. For what value(s) of k will the graphs of the functions $y = -2\sqrt{x+1}$ and $y = \sqrt{x-2} + k$ intersect? (Assume x and k are real numbers.)




- (b) Part of the graph for $y = f(x)$ is shown, $0 \leq x < 2$.
If $f(x+2) = \frac{1}{2}f(x)$ for all real values of x , draw the graph for the intervals, $-2 \leq x < 0$ and $2 \leq x < 6$.




8.  (a) The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a . Prove that each of these parabolas pass through a common point and determine the coordinates of this point.



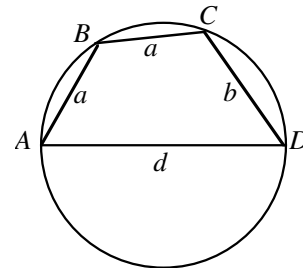
- (b) The vertices of the parabolas in part (a) lie on a curve. Prove that this curve is itself a parabola whose vertex is the common point found in part (a).

9.  A 'millennium' series is any series of consecutive integers with a sum of 2000. Let m represent the first term of a 'millennium' series.

- (a) Determine the minimum value of m .
(b) Determine the smallest possible positive value of m .

10.  $ABCD$ is a cyclic quadrilateral, as shown, with side $AD = d$, where d is the diameter of the circle. $AB = a$, $BC = a$ and $CD = b$. If a , b and d are integers $a \neq b$,

- (a) prove that d cannot be a prime number.
(b) determine the *minimum* value of d .





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Company

Northern Telecom
(Nortel)

Manulife
Financial

Equitable Life
of Canada


Time: $2\frac{1}{2}$ hours

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Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 2 marks each (questions 1-2) or 3 marks each (questions 3-7). **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.


Instructions for SHORT ANSWER parts


1. **SHORT ANSWER** parts are indicated like this: 
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.


Instructions for FULL SOLUTION parts


1. **FULL SOLUTION** parts are indicated like this: 
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.


NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.


- NOTE:
- Please read the instructions on the front cover of this booklet.
 - Place all answers in the answer booklet provided.
 - For questions marked “”, full marks will be given for a correct answer which is placed in the box in the answer booklet. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.
 - It is expected that all calculations and answers will be expressed as exact numbers such as 4π , $2 + \sqrt{7}$, etc., except where otherwise indicated.


1.  (a) If one root of $x^2 + 2x - c = 0$ is $x = 1$, what is the value of c ?

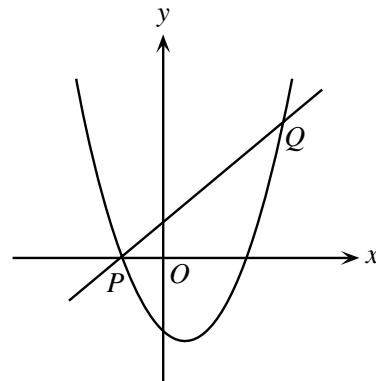
 (b) If $2^{2x-4} = 8$, what is the value of x ?

 (c) Two perpendicular lines with x -intercepts -2 and 8 intersect at $(0, b)$. Determine all values of b .


2.  (a) The vertex of $y = (x-1)^2 + b$ has coordinates $(1, 3)$. What is the y -intercept of this parabola?


 (b) What is the area of $\triangle ABC$ with vertices $A(-3, 1)$, $B(5, 1)$ and $C(8, 7)$?


 (c) In the diagram, the line $y = x + 1$ intersects the parabola $y = x^2 - 3x - 4$ at the points P and Q . Determine the coordinates of P and Q .




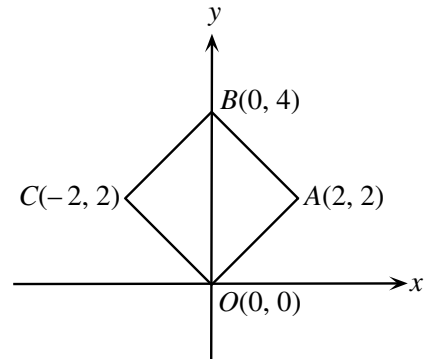
3.  (a) The graph of $y = m^x$ passes through the points $(2, 5)$ and $(5, n)$. What is the value of mn ?


 (b) Jane bought 100 shares of stock at \$10.00 per share. When the shares increased to a value of $\$N$ each, she made a charitable donation of all the shares to the Euclid Foundation. She received a tax refund of 60% on the total value of her donation. However, she had to pay a tax of 20% on the increase in the value of the stock. Determine the value of N if the difference between her tax refund and the tax paid was \$1000.

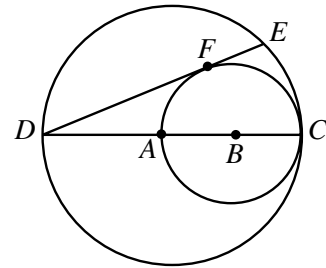
4.  (a) Consider the sequence $t_1 = 1$, $t_2 = -1$ and $t_n = \left(\frac{n-3}{n-1}\right)t_{n-2}$ where $n \geq 3$. What is the value of t_{1998} ?


 (b) The n th term of an arithmetic sequence is given by $t_n = 555 - 7n$. If $S_n = t_1 + t_2 + \dots + t_n$, determine the smallest value of n for which $S_n < 0$.

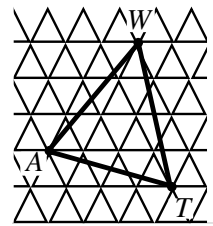
5.  (a) A square $OABC$ is drawn with vertices as shown. Find the equation of the circle with largest area that can be drawn inside the square.




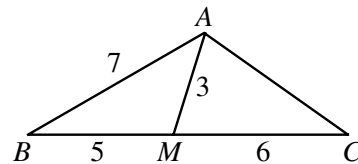
-  (b) In the diagram, DC is a diameter of the larger circle centred at A , and AC is a diameter of the smaller circle centred at B . If DE is tangent to the smaller circle at F , and $DC = 12$, determine the length of DE .



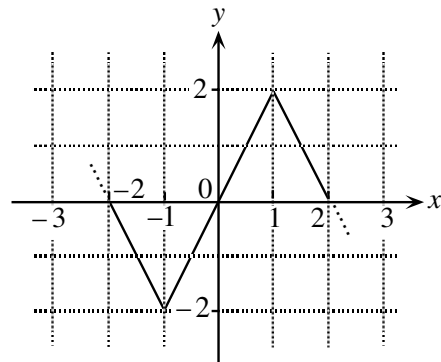
6.  (a) In the grid, each small equilateral triangle has side length 1. If the vertices of $\triangle WAT$ are themselves vertices of small equilateral triangles, what is the area of $\triangle WAT$?



-  (b) In $\triangle ABC$, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC .




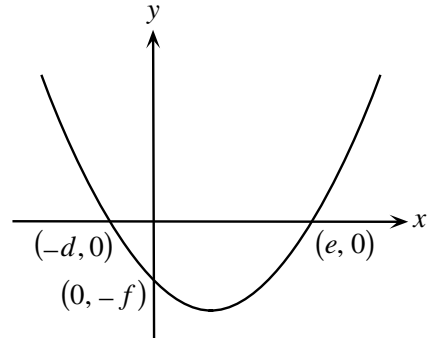
7. (a) The function $f(x)$ has period 4. The graph of one period of $y = f(x)$ is shown in the diagram. Sketch the graph of $y = \frac{1}{2}[f(x-1) + f(x+3)]$, for $-2 \leq x \leq 2$.




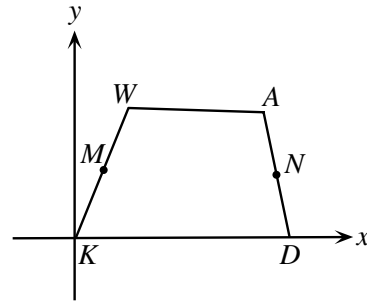
-  (b) If x and y are real numbers, determine all solutions (x, y) of the system of equations


$$\begin{aligned} x^2 - xy + 8 &= 0 \\ x^2 - 8x + y &= 0. \end{aligned}$$

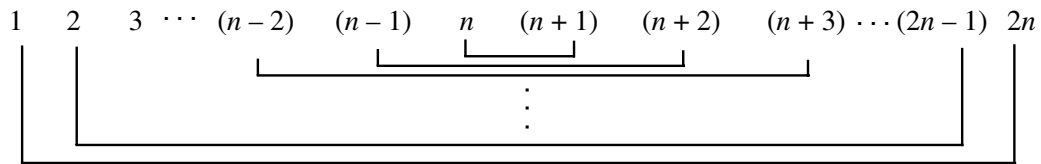
8.  (a) In the graph, the parabola $y = x^2$ has been translated to the position shown. Prove that $de = f$.




-  (b) In quadrilateral $KWAD$, the midpoints of KW and AD are M and N respectively. If $MN = \frac{1}{2}(AW + DK)$, prove that WA is parallel to KD .



9.  Consider the first $2n$ natural numbers. Pair off the numbers, as shown, and multiply the two members of each pair. Prove that there is no value of n for which two of the n products are equal.



10.  The equations $x^2 + 5x + 6 = 0$ and $x^2 + 5x - 6 = 0$ **each** have integer solutions whereas only one of the equations in the pair $x^2 + 4x + 5 = 0$ and $x^2 + 4x - 5 = 0$ has integer solutions.
- (a) Show that if $x^2 + px + q = 0$ and $x^2 + px - q = 0$ **both** have integer solutions, then it is possible to find integers a and b such that $p^2 = a^2 + b^2$. (i.e. (a, b, p) is a Pythagorean triple).
- (b) Determine q in terms of a and b .